Problem Set 1

DUE: In class Thursday, Sept. 20. *Late papers will be accepted until 1:00 PM Friday.*

REMARK: It is often effective to work together with someone else.

1. Comment on the following proof that there is no uninteresting positive integer.

   Say there were one. Then let N be the smallest uninteresting integer. Then it would be interesting because it is the smallest uninteresting integer.

2. All odd numbers have the property that when they are divided by 4 their remainder is either 1 or 3. Thus the are either of the form $4k+1$ or $4k-1$, where k is an integer. Prove that there are an infinite number of primes of the form $4k-1$.

   [SUGGESTION:] Following the idea in Euclid's proof that there are an infinite number of primes, let p be a prime and let $N := 2^2 \cdot 3 \cdot 5 \cdot 7 \cdots p - 1$. Then N has the form $4k-1$ and is not divisible by any of the primes up to p. Use the observation that the product of any two numbers of the form $4n+1$ also has this form to show that N cannot be a product of primes only of the form $4n+1$.]

3. a) Prove that $\sin x$ is not a polynomial.

   b) Prove that $\sin x$ is not a rational function, that is, does not have the form $\dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.

   c) Prove that the function e^x is not a polynomial.

4. The number e is defined as

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots .$$

   Prove that e is not a rational number by the following steps.

   a) Show that $2 < e < 3$. So e is definitely not an integer.

   b) By contradiction, say $e = \frac{p}{q}$, where p and q are positive integers with $q \ge 2$. Show that

$$e\, q! = N + \frac{c}{q+1},$$

   where N is an integer and $0 < c < e$. Thus, conclude that $\frac{c}{q+1}$ must be an integer.

   c) Then show that this contradicts $e < 3$ and $q + 1 \ge 3$.

5. In class we discussed the problem "Square of Differences". Investigate the same problem, only this time replacing the square by a triangle and a pentagon.

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6. Can you draw a continuous curve that passes through each of the sides of the figure (on the left) exactly once? On the right is an unsuccessful attempt (it misses the middle side on the bottom).