Math 210, Spring 2019 — Jerry L. Kazdan

Problem Set 6

DUE: In class Thurs. Mar. 21 [*Late papers will be accepted until 1:00 PM Fri*].

Problems

1. Use the Method of Least Squares to find the straight line $y = ax + b$ that best fits the following data given by the following four points (x_j, y_j), $j = 1, \ldots, 4$:

$$(-2, 4), \qquad (-1, 3), \qquad (0, 1), \qquad (2, 0).$$

Ideally, you'd like to pick the coefficients a and b so that the four equations $ax_j + b = y_j$, $j = 1, \ldots, 4$ are all satisfied. Since this probably can't be done, one uses least squares to find the best possible a and b.

2. Find a curve of the form $y = a + bx + cx^2$ that best fits the following data

x	-2	-1	0	1	2	3	4
y	4	1.1	-0.5	1.0	4.3	8.1	17.5

3. Find a plane of the form $z = ax + by + c$ that best fits the following data

x	0	1	0	1	0
y	0	1	1	0	-1
z	1.1	2	-0.1	3	2.2

4. The water level in the North Sea is mainly determined by the so-called M2 tide, whose period is about 12 hours. The height $H(t)$ thus roughly has the form

$$H(t) = c + a\sin(2\pi t/12) + b\cos(2\pi t/12),$$

where time t is measured in hours (note $\sin(2\pi t/12$ and $\cos(2\pi t/12)$ are periodic with period 12 hours). Say one has the following measurements:

t (hours)	0	2	4	6	8	10
$H(t)$ (meters)	1.0	1.6	1.4	0.6	0.2	0.8

Use the method of least squares with these measurements to find the constants a, b, and c in $H(t)$ for this data.

5. a) Some experimental data $(x_i,\, y_i)$ is believed to fit a curve of the form

$$y = \frac{1+x}{a+bx^2},$$

where the parameters a and b are to be determined from the data. The method of least squares does not apply directly to this since the parameters a and b do not appear linearly. Show how to find a modified equation to which the method of least squares does apply.

b) Repeat part a) for the curve $y = ax^b$ (this assumes that $x \ge 0$).

c) Repeat part a) for the curve $y = 1 - e^{-ax^b}$ (this assumes that $x \ge 0$) and implies that $y < 1$).

Bonus Problem

[Please give solutions directly to Professor Kazdan]

1-B [Principal Component Analysis] Let $Z_j = (x_j, y_j)$, $j = 1, \ldots, , N$ be (data) points in the plane $\mathbb{R}^2$, say the height and weight of the j^{th} person in a medical test. Problem: find the straight line $\mathcal{L} := \{(x,y) \in \mathbb{R}^2 \,|\, ax+by=c\}$ that best fits this data in the sense that it minimizes the function

$$Q(\mathcal{L}) := \sum_{j=1}^{N} [\text{Distance}(Z_j, \mathcal{L})]^2.$$

Remark: An equivalent way to write a line $\mathcal{L}$ in the plane, $\mathbb{R}^2$ is

$$\mathcal{L} = \{X \in \mathbb{R}^2 \,|\, X = X_0 + tV\},$$

where $V \in \mathbb{R}^2$ is a unit vector and $X_0 \perp V \in \mathbb{R}^2$ is a specified point on the line. Use either $ax+by=c$ or this, whichever you prefer.

a) Thus, we need to determine the parameters a, b, and c. As should be clear in your computation, it is simplest to investigate first the special case where $\sum_{j=1}^{N} Z_j = 0$.

b) Apply this procedure to the data points $(0,0)$, $(1,3)$, and $(2,7)$.

2-B a) Give an example of a 3×3 anti-symmetric matrix.

b) If A is any anti-symmetric matrix, show that $\langle X,\, AX\rangle = 0$ for all vectors X.

c) Say $X(t)$ is a solution of the differential equation $\dfrac{dX}{dt} = AX$, where A is an *anti-symmetric* matrix. Show that $\|X(t)\| = \text{constant}$. [Hint: Compute the derivative of $\|X(t)\|^2$.]

3-B Let $A : \mathbb{R}^n \to \mathbb{R}^k$ be a linear map. If A is not one-to-one, but the equation $Ax = y$ has some solution, then it has many. Is there a "best" possible answer? What can one say? Think about this before reading the next paragraph.

If there is some solution of $Ax = y$, show there is exactly one solution x_1 of the form $x_1 = A^*w$ for some w, so $AA^*w = y$. Moreover of all the solutions x of $Ax = y$, show that x_1 is closest to the origin (in the Euclidean distance). [REMARK: This situation is related to the case where where A is not onto, so there may not be a solution — but the method of least squares gives an "best" approximation to a solution.]

[Last revised: March 14, 2019]