Problem Set 5

DUE: In class Tues. Feb. 26 [*Late papers will be accepted until 1:00 PM Wed*].

Note that Exam 1 will be on Tuesday, March 12 in class from 1:30–2:50. Closed book but you may use one 3×5 card with notes on both sides.

Problems

REMARK: The last problem concerns the basic mathematics of making a loan, say to buy a car. It involves only high school algebra. Everyone in this course should understand it thoroughly. It is not just pushing a button on a calculator.

1. a) For which values of the constant a and b are the vectors $U = (1 + a, -2b, 4)$ and $V = (2, 1, -1)$ perpendicular?

   b) For which values of the constant a, and b is the above vector U, perpendicular to both V and the vector $W = (1, 1, 0)$?

2. Let $X = (3, 4, 0)$ and $Y = (1, -1, 1)$.

   a) Write the vector Y in the form $Y = cX + V$, where V is orthogonal to X. Thus, you need to find the constant c and the vector V. Interpretation: You are decomposing Y as a sum of vectors, one in the direction of X and one perpendicular to X.

   b) Compute $\|X\|$, $\|Y\|$, and $\|V\|$ and verify the Pythagorean relation
   $$\|Y\|^2 = \|cX\|^2 + \|V\|^2.$$

3. If a vector X is written as $X = aU + bV$, where U and V are non-zero orthogonal vectors, show that $a = \langle X, U\rangle/\|U\|^2$ and $b = \langle X, V\rangle/\|V\|^2$.

4. The origin and the vectors X, Y, and $X + Y$ define a parallelogram whose diagonals have length $X + Y$ and $X - Y$. Prove the *parallelogram law*
   $$\|X + Y\|^2 + \|X - Y\|^2 = 2\|X\|^2 + 2\|Y\|^2;$$
   This states that in a parallelogram, the sum of the squares of the lengths of the diagonals equals the sum of the squares of the four sides.

5. a) If a certain matrix C satisfies $\langle X, CY\rangle = 0$ for *all* vectors X and Y, show that $C = 0$.

   b) If the matrices A and B satisfy $\langle X, AY\rangle = \langle X, BY\rangle$ for all vectors X and Y, show that $A = B$.

6. a) Find the distance from the point $(2, -1)$ to the straight line $3x - 4y = 0$.

   b) Find the distance from the straight line $3x - 4y = 10$ to the origin.

   c) Find the distance from the straight line $ax + by = c$ to the origin.

   d) Find the distance between the parallel lines $ax + by = c$ and $ax + by = \gamma$.

   e) Find the distance from the plane $ax + by + cz = d$ to the origin.

7. The equation of a straight line in $\mathbb{R}^3$ can be written as $X(t) = X_0 + tV$, $-\infty < t < \infty$, where X_0 is a point on the line and $V \neq 0$ is a vector along the line (in a physical setting, V might be the *velocity* vector).

   a) Find the distance from the origin to this line.

   b) Find the distance from a point $Z \in \mathbb{R}^3$ to this line.

   c) Let $Y(s) = Y_0 + sW$, $-\infty < s < \infty$, be another straight line with W not a multiple of V (so the lines are not parallel). If the lines don't intersect, find the distance between them.

      [SUGGESTION: Let $h(s,t) := \|X(t) - Y(s)\|^2$ and determine where h has its minimum.]

8. Let P_1, P_2, ..., P_k be points in $\mathbb{R}^n$. For $X \in \mathbb{R}^n$ let

$$Q(X) := \|X - P_1\|^2 + \|X - P_2\|^2 + \cdots \|X - P_k\|^2.$$

   Determine the point X_0 that minimizes $Q(X)$.

   [SUGGESTION: Let $V \in \mathbb{R}^n$ be *any* vector, t any real number, and $h(t) := Q(X_0 + tV)$. Then $h(t)$ has a minimum at $t = 0$ (why?) so $h'(0) = 0$ for any V.]

9. a) If X and Y are real vectors, show that

$$\langle X, Y \rangle = \frac{1}{4}\left(\|X + Y\|^2 - \|X - Y\|^2\right).$$

   This formula is the simplest way to recover properties of the inner product from the norm.

   b) As an application, show that if a square matrix R has the property that it preserves length (rigid motion), so $\|RX\| = \|X\|$ for every vector X, then it preserves the inner product, that is, $\langle RX, RY \rangle = \langle X, Y \rangle$ for all vectors X and Y.

   `https://www.mathworks.com/content/dam/mathworks/mathworks-dot-com/moler/eigs.pdf`, page 34, connected rigid motions

   See also: `https://www.biomotionlab.ca/demos/`

10. [MAKING A LOAN] Susan borrows P_0 dollars to buy a car. The annual interest rate is i (perhaps 6%, 7%, etc.) compounded monthly, so at the end of the first month she owes $[1+(i/12)]P_0$ dollars, where here i is written as, say, $.06$, etc.. She will repay the loan with equal monthly payments of M dollars. Thus, just *after* she has made the first monthly payment she owes

$$P_1 = [1+(i/12)]P_0 - M$$

dollars. One month later, she owes $[1+(i/12)]P_1$ dollars so just after she has made the second monthly payment she owes

$$P_2 = [1+(i/12)]P_1 - M$$

and so on. One can also write this as

$$M = (P_1 - P_2) + I_2$$

where $P_1 - P_2$ in the decrease in the amount owed and $I_2 = (i/12)P_1$ is the interest component of the second monthly payment.

a) What is the minimum monthly payment M if her balance is to *decrease*?

b) Write the formula for P_2, the amount she owes just after making the second monthly payment, in terms of P_0 and M. [REMARK: I suggest letting $c = 1+(i/12)$. It makes the computation more transparent.]

c) How much, P_k, does she owe just after making the k^{th} monthly payment? Write your answer in terms of P_0, M, and k. You may find it useful to recall the formula for the sum of a *geometric series*: $1+r+r^2+\cdots+r^n$.

d) For tax records, find a formula for the interest component of the k^{th} monthly payment $I_k = (i/12)P_{k-1}$. Your formula should involve only P_0, M, i (or c), and k.

e) How many monthly payments, N, are needed until the loan is completely repaid? [Hint: take logarithms].

f) If the loan is to be repaid in exactly N monthly payments, how much, M, should she repay each month?

[Last revised: February 28, 2019]