Math 210, Spring 2019 Jerry L. Kazdan

Problem Set 4

DUE: In class Tues. Feb. 19 [*Late papers will be accepted until 1:00 PM Wed*].

Lots of problems this week. Fortunately a number of them are short – but don't wait until Monday night!

Most of these problems should be a review of the basic linear algebra of Math 240, but emphasizing thinking of a system of linear equations as a linear mapping. They should be *very* short. In class on we'll discuss this more.

Problems

1. If A is a 5×5 matrix with $\det A = -1$, compute $\det(-2A)$.

2. Consider the system of equations

$$\begin{aligned} x+y-\ z &= a \\ x-y+2z &= b. \end{aligned}$$

   a) Find the general solution of the homogeneous equation, so $a=b=0$.
   b) A particular solution of the inhomogeneous equations when $a=1$ and $b=2$ is $x=1$, $y=1$, $z=1$. Find the most general solution of the inhomogeneous equations.
   c) Find some particular solution of the inhomogeneous equations when $a=-1$ and $b=-2$.
   d) Find some particular solution of the inhomogeneous equations when $a=3$ and $b=6$.

   [Remark: After you have done part a), it is possible immediately to write the solutions to the remaining parts.]

3. Solve the given system – or show that no solution exists:

$$\begin{aligned} x+2y &= 1 \\ 3x+2y+4z &= 7 \\ -2x+y-2z &= -1 \end{aligned}$$

4. Say you have k linear algebraic equations in n variables; in matrix form we write $AX=Y$. Give a proof or counterexample for each of the following.
   a) If $n=k$ there is always *at most one* solution.
   b) If $n>k$ you can *always* solve $AX=Y$.

c) If $n > k$ the homogeneous equation $AX = 0$ has at least one solution $X \neq 0$..

d) If $n < k$ then for *some* Y there is *no* solution of $AX = Y$.

e) If $n < k$ the *only* solution of $AX = 0$ is $X = 0$.

5. Let $A : \mathbb{R}^n \to \mathbb{R}^k$ be a real matrix, not necessarily square. If two rows of A are the same, show that A is not onto by finding a vector $y = (y_1, \ldots, y_k)$ that is not in the image of A. [HINT: This is a mental computation if you write out the equations $Ax = y$ explicitly.]

6. Let $A : \mathbb{R}^n \to \mathbb{R}^k$ be a real matrix, not necessarily square. If two columns of A are the same, show that A is not one-to-one by finding a vector $x = (x_1, \ldots, x_n)$ that satisfies $Ax = 0$.

7. The following 2×2 matrices are valuable examples that may be surprising[1]:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad C = PR = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Geometrically, P is an orthogonal projection onto the x_1 axis, that is, if $X = (x_1,\, x_2) \in \mathbb{R}^2$ is a (column) vector in the plane, then PX is its orthogonal projection onto the x_1 axis. Similarly, R is a rotation by 90 degrees clockwise.

Compute (and interpret geometrically):

$$P^2, \quad P^3, \quad R^2, \quad R^3, \quad R^4, \quad PR, \quad RP, \quad C^2, \quad CP, \quad PC.$$

8. Let A and B be $n \times n$ matrices with $AB = 0$. Give a proof or counterexample for each of the following.

a) Either $A = 0$ or $B = 0$ (or both).

b) $BA = 0$

c) If $\det A = -3$, then $B = 0$.

d) If B is invertible then $A = 0$.

e) There is a vector $V \neq 0$ such that $BAV = 0$.

9. Let A be a 4×4 matrix with determinant 7. Give a proof or counterexample for each of the following.

a) For some vector $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution.

[1] The computer graphics examples in `https://www.math.upenn.edu/~kazdan/320F18/notes/Maple/F1.pdf` may also be illuminating.

b) For some vector $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

c) For some vector $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has no solution.

d) For all vectors $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution.

10. a) Find a 2×2 matrix that rotates the plane by $+45$ degrees ($+45$ degrees means 45 degrees *counterclockwise*).

    b) Find a 2×2 matrix that rotates the plane by $+45$ degrees followed by a reflection across the horizontal axis.

    c) Find a 2×2 matrix that reflects across the horizontal axis followed by a rotation the plane by $+45$ degrees.

    d) Find a matrix that rotates the plane through $+60$ degrees, keeping the origin fixed.

    e) Find the inverse of each of these maps.

11. Find a real 2×2 matrix A (other than $A = I$) such that $A^5 = I$.

12. Proof or counterexample. In these L is a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$, so its representation will be as a 2×2 matrix.

    a) If L is invertible, then L^{-1} is also invertible.

    b) If $LV = 5V$ for all vectors V, then $L^{-1}W = (1/5)W$ for all vectors W.

    c) If L is a rotation of the plane by 45 degrees *counterclockwise*, then L^{-1} is a rotation by 45 degrees *clockwise*.

    d) If L is a rotation of the plane by 45 degrees counterclockwise, then L^{-1} is a rotation by 315 degrees counterclockwise.

    e) The zero map ($0\mathbf{V} = 0$ for all vectors $\mathbf{V}$) is invertible.

    f) The identity map ($I\mathbf{V} = \mathbf{V}$ for all vectors $\mathbf{V}$) is in vertible.

    g) If L is invertible, then $L^{-1}0 = 0$.

    h) If $L\mathbf{V} = 0$ for some non-zero vector $\mathbf{V}$, then L is not invertible.

    i) The identity map (say from the plane to the plane) is the only linear map that is its own inverse: $L = L^{-1}$.

13. Let A be a matrix, not necessarily square. Say $\mathbf{V}$ and $\mathbf{W}$ are particular solutions of the equations $A\mathbf{V} = \mathbf{Y}_1$ and $A\mathbf{W} = \mathbf{Y}_2$, respectively, while $\mathbf{Z} \neq 0$ is a solution of the homogeneous equation $A\mathbf{Z} = 0$. Answer the following in terms of $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{Z}$.

    a) Find some solution of $A\mathbf{X} = 3\mathbf{Y}_1$.

    b) Find some solution of $A\mathbf{X} = -5\mathbf{Y}_2$.

    c) Find some solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$.

d) Find another solution (other than $\mathbf{Z}$ and 0) of the homogeneous equation $A\mathbf{X} = 0$.

e) Find *two* solutions of $A\mathbf{X} = \mathbf{Y}_1$.

f) Find another solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$.

g) If A is a square matrix, then $\det A = ?$

h) If A is a square matrix, for any given vector $\mathbf{W}$ can one always find at least one solution of $A\mathbf{X} = \mathbf{W}$? Why?

14. Let R, M, and N be linear maps from the (two dimensional) plane to the plane given in terms of the standard $\mathbf{i}$, $\mathbf{j}$ basis vectors by:

$$R\mathbf{i} = \mathbf{j}, \quad R\mathbf{j} = -\mathbf{i} \qquad M\mathbf{i} = -\mathbf{i}, \quad M\mathbf{j} = \mathbf{j} \qquad N\mathbf{v} = -\mathbf{v} \text{ for } \textit{all} \text{ vectors } \mathbf{v}$$

a) Describe (pictures?) the actions of the maps R, R^2, R^{-1}, M, M^2, M^{-1} and N.

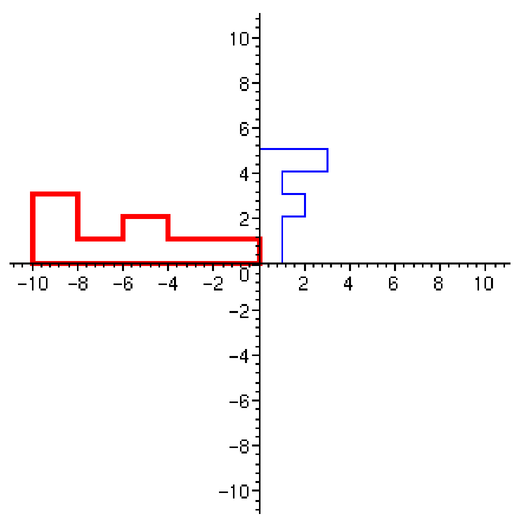
b) Describe the actions of the maps RM, MR, RN, NR, MN, and NM [here we use the standard convention that the map RM means *first* use M *then* R]. Which pairs of these maps commute?

c) Which of the following identities are correct—and why?

1) $R^2 = N$	2) $N^2 = I$	3) $R^4 = I$	4) $R^5 = R$
5) $M^2 = I$	6) $M^3 = M$	7) $MNM = N$	8) $NMN = R$

d) Find matrices representing each of the maps R, R^2, R^{-1}, M, and N.

15. a). Find a linear map of the plane, $A : \mathbb{R}^2 \to \mathbb{R}^2$ that does the following transformation of the letter **F** (here the smaller **F** is transformed to the larger one):



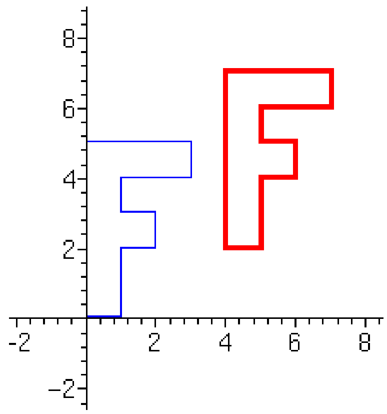
b). Find a linear map of the plane that inverts this map, that is, it maps the larger **F** to the smaller.

16. Linear maps $F(X) = AX$, where A is a matrix, have the property that $F(0) = A0 = 0$, so they necessarily leave the origin fixed. It is simple to extend this to include a translation,

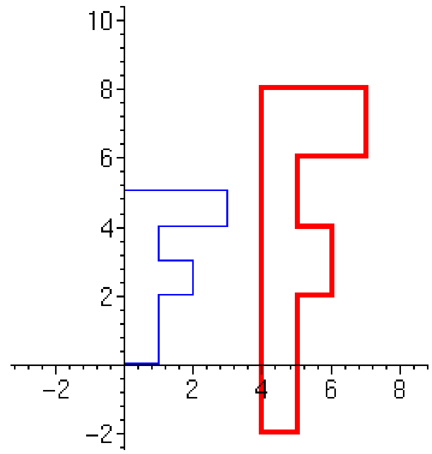
$$F(X) = V + AX,$$

where V is a vector. Note that $F(0) = V$.

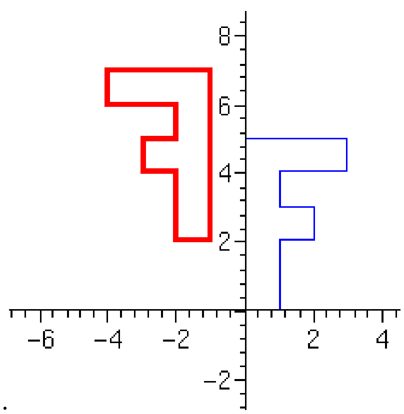
Find the vector V and the matrix A that describe each of the following mappings [here the light blue F is mapped to the dark red F].



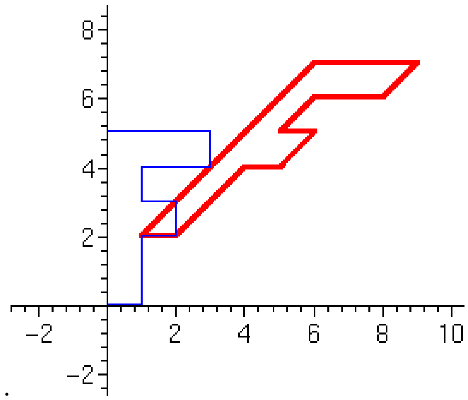
a).



b).



c).



d).

Bonus Problem

[Please give solutions directly to Professor Kazdan]

1-B [How to Rotate a Mattress]. It is standard to rotate a mattress so that it wears more evenly. For this task, one needs to understand the symmetries of a mattress – which is just a rectangular box whose height, width, and length are distinct.

[As a warm-up, understand all the symmetries of a square.]

[Last revised: February 12, 2019]