Problem Set 3

DUE: In class Tues. Feb. 12 [*Late papers will be accepted until 1:00 PM Wed*].

1. Say you pick a point x at random in the interval $0 \le x \le 1$. What is the probbility that it will be in the set

   a) $S = \{x \le 1/2 \text{ and } x \ge 1/4\}$?

   b) $T = \{x \le 1/2 \text{ and } x \ge 3/4\}$?

2. Say you pick a point (x, y) at random in the unit square $0 \le x \le 1$, $0 \le y \le 1$. Compute the probability it will bw in the set

$$S = \{\ x \le y, \quad \text{and} \quad x \le 1/2, \quad \text{and} \quad y - x \le 1/2, \quad \text{and} \quad 1 - y \le 1/2\ \}.$$

3. Let T be the transition matrix of a Markov process. If P is a probability vector, show that TP is also a probability vector.

4. A long queue in front of a Moscow market in the Stalin era sees the butcher whisper to the first in line. He tells her "Yes, there is steak today." She tells the one behind her and so on down the line. However, Moscow housewives are not reliable transmitters. If one is told "yes", there is only an 80% chance she'll report "yes" to the person behind her. On the other hand, being optimistic, if one hears "no", she will report "yes" 40% of the time.

   If the queue is very long, what fraction of them will hear "there is no steak"? [This problem can be solved without finding a formula for P^n (here P is the transition matrix) – although you might find it a challenge to find the formula].

5. Suppose there is an epidemic in which every month half of those who are well become sick, half of those who are sick get well, and a quarter of those who are sick die. Find the steady state for the corresponding Markov process

$$\begin{pmatrix} w_{n+1} \\ s_{n+1} \\ d_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 1 \end{pmatrix} \begin{pmatrix} w_n \\ s_n \\ d_n \end{pmatrix}$$

6. Multinational companies in the Americas, Asia, and Europe have assets of \$4 trillion. At the start, \$2 trillion are in the Americas and \$2 trillion are in Europe. Each year 1/2 of the Americas money stays home and 1/4 goes to each of Asia and Europe. For Asia and Europe, 1/2 stays home and 1/2 is sent to the Americas.

a) Let C_k be the column vector with the assets of the Americas, Asia, and Europe at the beginning of year k. Find the transition matrix T that gives the amount in year $k+1$: $C_{k+1} = TC_k$

b) Find the eigenvalues and eigenvectors of T.

c) Find the limiting distribution of the \$4 trillion as the world ends

d) Find the distribution of the \$4 trillion at year k.

7. A certain plant species has either red (dominant), pink (hybred), or white (recesive) flowers, depending on its genotype. If you cross a pink plant with any other plant, the probability distribution of the offsprings are prescribed by the transition matrix

$$T := \begin{pmatrix} .5 & .25 & 0 \\ .5 & .5 & .5 \\ 0 & .25 & .5 \end{pmatrix}.$$

The first column of T means that if you cross a red with a pink, then 50% of the time you'll get a red and 50% a pink – and never get a white. The second column gives the result if you cross a pink with a pink (25% red, 50% pink, 25% white) while the third column concerns crossing a white with a pink.

In the long run, if you continue crossing the offsprings with only pink plants, what percentage of the three types of flowers would you expect to see in your garden?

8. Assume (naively) that a person's job can be classified as professional, skilled, or unskilled. Assume that, of the children of professional parents, 80 percent are professional, 10 percent are skilled, and 10 percent are unskilled. In the case of children of skilled, 60 percent are skilled, 20 percent are professional, and 20 percent are unskilled. Finally, in the case of unskilled, 50 percent of the children are unskilled, and 25 percent each are in the other two categories. Assume that every family has at least one child.

a) Form a Markov chain by following the job of a randomly chosen child of a given family through several generations. Set up the matrix of transition probabilities.

b) Find the probability that a randomly chosen grandchild of an unskilled worker is a professional.

9. If T is the transition matrix of a *regular* Markov process (so for some k all the entries of T^k are positive), we know there is a probability vector P_∞ so that if P_0 is any initial probability vector, then $\lim_{k\to\infty} T^k P_0 = P_\infty$.

Show that the matrix $\lim_{k\to\infty} T^k$ has *all* of its columns equal to P_∞.

10. Let T be the transition matrix for a Markov chain. If there is some power k for which all of the elements of T^k are positive, show that all the elements of T^{k+1} are positive.

Bonus Problems

[Please give solutions directly to Professor Kazdan]

1-B Three friends sit around a table, each with a large plate of cheese. Instead of eating it, every minute each of them simultaneously pass half of their cheese to the neighbor on the left and the other half to the neighbor on the right.

a) Is it true that the amount of cheese on the first person's plate will converge to some limit as time goes to infinity? Explain.

b) The next week they meet again, adding a fourth friend and follow the same procedure. What can you say about the eventual distribution of the cheese?

2-B Snow White distributed 21 liters of milk among the seven dwarfs. The first dwarf then distributed the contents of his pail evenly to the pails of other six dwarfs. Then the second did the same, and so on. After the seventh dwarf distributed the contents of his pail evenly to the other six dwarfs, it was found that each dwarf had exactly as much milk in his pail as at the start.

What was the initial distribution of the milk?

[From: K. Splinder *Abstract Algebra with Applications*, Vol. 1, page 192, Dekker (1994)]

[Last revised: March 16, 2019]