Waitin' for the bus outta here

In a certain town there are 2 bus companies whose buses stop at the Main Street station. The first company's bus arrives at this station every k_1 minutes, the second every k_2 minuters (we assume $k_1 \le k_2$) but the times of arrival of the previous busses are unknown. The question is, what is the average length of time that you will wait for a bus after arriving at the station?

The solution

Let the random variable X_i denote the time between your arrival at the station and arrival of the next bus of the ith company. The cumulative distribution function (cdf) of X_i is

$$F_i(t) \stackrel{\text{def}}{=} \text{Prob}(X_i \le t) = \begin{cases} 0, & \text{if } t < 0; \\ t/k_i, & \text{if } 0 \le t \le k_i; \\ 1, & \text{if } t > k_i. \end{cases}$$

The probability that *all* of the X_i's are $\ge t$ is $(1 - F_1(t))(1 - F_2(t))$ which is 0 for $t > k_1$, reflecting the fact that you will surely not wait more than k_1 minutes. Hence the probability that at least one of the X_i's is $\le t$

$$1 - (1 - F_1(t))(1 - F_2(t))$$

The waiting time that we actually experience at the bus stop will be $X = \min_i X_i$, and the cdf of this random variable is

$$F(t) \stackrel{\text{def}}{=} \text{Prob}(X \le t) = \begin{cases} 0, & \text{if } t < 0; \\ 1 - \left(1 - \frac{t}{k_1}\right)\left(1 - \frac{t}{k_2}\right), & \text{if } 0 < t < k_1; \\ 1, & \text{if } t > k_1. \end{cases}$$

Thus the density function, $f(t) = F'(t)$ is

$$f(t) = \begin{cases} 0, & \text{if } t < 0; \\ \left(\frac{1}{k_1} + \frac{1}{k_2}\right) - \frac{2t}{k_1 k_2}, & \text{if } 0 \le t \le k_1; \\ 0, & \text{if } t > k_1. \end{cases}$$

Finally the expected waiting time is

$$\begin{aligned} E(\mathbf{k}) = \int_{-\infty}^{\infty} t\, f(t)\, dt = \int_0^{k_1} \left[\left(\frac{1}{k_1} + \frac{1}{k_2}\right) t - \frac{2t^2}{k_1 k_2}\right] dt \\ = \frac{k_1}{2} - \frac{k_1^2}{6k_2} \end{aligned} \tag{1}$$

In particular, if $k_1 = k_2 = 10$ minutes, then the expected waiting time is $k_1/3 = 10/3$ minutes.