INSIGHTS PUZZLE

Why *e*, the Transcendental Math Constant, Is Just the Best

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The solution to our puzzle about Euler's number explains why e *pops up in situations that involve optimality.*

James Round for Quanta Magazine

Last month, we presented three puzzles that seemed ordinary enough but contained a numerical twist. Hidden below the surface was the mysterious transcendental number *e*. Most familiar as the base of natural logarithms, Euler's number *e* is a universal constant with an infinite decimal expansion that begins with 2.7 1828 1828 45 90 45... (spaces added to highlight the quasi-

n	Expected empty seats E	Value	1st Diff. Seq. D	Value	2nd Diff. Seq. DD	Value
0	E_0	0	D_0	0	DD_0	0
1	E_1	1	$D_1 = E_1 - E_0$	1	$DD_1 = D_1 - D_0$	1
2	E_2	0	$D_2 = E_2 - E_1$	-1	$DD_2 = D_2 - D_1$	-2
3	E_3	1	$D_3 = E_3 - E_2$	1	$DD_3 = D_3 - D_2$	2
4	E_4	$\frac{2}{3}$	$D_4 = E_4 - E_3$	$-\frac{1}{3}$	$DD_4 = D_4 - D_3$	$-\frac{4}{3}$
5	E_5	1	$D_5 = E_5 - E_4$	$\frac{1}{3}$	$DD_5 = D_5 - D_4$	$\frac{2}{3}$
6	E_6	$\frac{16}{15}$	$D_6 = E_6 - E_5$	$\frac{1}{15}$	$DD_6 = D_6 - D_5$	$-\frac{4}{15}$
7	E_7	$\frac{11}{9}$	$D_7 = E_7 - E_6$	$\frac{7}{45}$	$DD_7 = D_7 - D_6$	$\frac{4}{45}$

By doing some tangled algebra on the original recurrence relation, we can derive a closed-form expression for DD_n as follows ($n \geq 1$):

$$DD_n = \frac{(-2)^{n-1}}{(n-1)!}$$

We can verify that this formula works by comparing it with the above table:

$DD_1 = \frac{(-2)^0}{0!} = 1$
$DD_2 = \frac{(-2)^1}{1!} = -2$
$DD_3 = \frac{(-2)^2}{2!} = 2$
$DD_4 = \frac{(-2)^3}{3!} = -\frac{8}{6} = -\frac{4}{3}$
$DD_5 = \frac{(-2)^4}{4!} = \frac{16}{24} = \frac{2}{3}$
$DD_6 = \frac{(-2)^5}{5!} = -\frac{32}{120} = -\frac{4}{15}$

$DD_7 = \frac{(-2)^6}{6!} = \frac{64}{720} = \frac{4}{45}$

Now here’s the kicker: Notice what happens when you add up all the DDs.

$DD_1 + DD_2 + DD_3 + \ldots + DD_{n-1} + DD_n =$

$D_1 - D_0 + D_2 - D_1 + D_3 - D_2 + \ldots + D_{n-1} - D_{n-2} + D_n - D_{n-1} = D_n - 0 = D_n$.

All the other terms cancel, leaving only D_n. This technique of solving a recurrence is aptly called telescoping.

Now let’s plug the closed-form expressions back in for each of the DDs.

$$D_n = 1 - \frac{2}{1!} + \frac{2^2}{2!} - \frac{2^3}{3!} + \frac{2^4}{4!} - \frac{2^5}{5!} + \cdots \quad .$$

Look familiar? Compare this to the infinite series for e^{-x} above. In fact, for large n, this is exactly the formula for e^{-2} or $\frac{1}{e^2}$.

So the square of our transcendental constant has magically appeared in the denominator of the first difference sequence derived from the number of empty seats. It’s there because the process entails a summation of successive powers of −2 divided by the corresponding factorial, which is exactly the structure of the powers of *e*.

We still have some work to do to get back from the D sequence to the E sequence by a similar process to get E_n. This value, divided by n, will give us the fraction of seats that are empty. To get to the finish line, we need to do some more tangled algebra. Suffice it to say that e^{-2} remains in the final expression, which looks like this:

$$\frac{E_n}{n} = \frac{1}{e^2} + \frac{2}{e^2}\left(\frac{1}{n}\right) + \cdots$$

In the limit of large n, only the first term remains, giving the result we already found: $\frac{1}{e^2}$. Note that for the large but finite numbers that Michel Nizette listed, just the sum of the first two terms fits values for the unfilled fraction almost exactly.

I hope you enjoyed the heavy dose of transcendence from this most fascinating and fundamental constant. The *Quanta* Insights award for this month goes jointly to Michel Nizette, for clarity of exposition, and Lazar Ilic for the usual mathematical mastery. Congratulations to both!

(Lazar Ilic, you are clearly one of the best mathematicians to grace the comments sections of our puzzles. As this is the third time you have won this award, by my count, and we want to give others a chance too, this is the final award we will bestow on you. You have hereby entered the Insights Hall of Fame. I do hope you will keep contributing your deeply insightful comments. We look forward to them.)

See you next time for new Insights!