Markov Chains: How Google Works

EXAMPLE 1 Say every year 5% of the people in California move Elsewhere while 1% of the people livine Elsewhere move to California. Is there an equilibrium population distribution, and if so, what is it?

Let C_0 be the initial population of California and E_0 the initial population Elsewhere. The at the beginning of the next year the population is

$$\begin{aligned} C_1 &= .95C_0 + .01E_0 \\ E_1 &= .05C_0 + .99E_0 \end{aligned}$$

and at the beginning of the second year

$$\begin{aligned} C_2 &= .95C_1 + .01E_1 \\ E_2 &= .05C_1 + .99E_1 \end{aligned}$$

. Introduce the matrix T and probability vector P_k

$$T := \begin{pmatrix} .95 & .01 \\ .05 & .99 \end{pmatrix} \qquad P_k := \begin{pmatrix} C_k \\ E_k \end{pmatrix}, \quad k = 0,\, 1,\, 2, \ldots$$

P_k is a *probability vector* if its elements are between 0 and 1 and their sum is 1: $C_k + E_k = 1$
Then the above equations say $P_1 = TP_0, \;\; P_2 = TP_1$, etc. Thus $P_2 = T^2 P_0$, so $P_k = T^k P_{k-1}$, $k = 1,\, 2, \ldots$.
In an *equilibrium state* P we have $P = TP$, that is $(T - I)P = 0$. the equations are clearly

$$\begin{aligned} C &= .95C + .01E \\ E &= .05C + .99E \end{aligned}, \qquad \text{that is} \qquad \begin{aligned} -.05C + .01E &= 0 \\ .05C - .01E &= 0 \end{aligned}.$$

along with $C + E = 1$. Solving these we find $C = \frac{1}{6}$ and $E = \frac{5}{6}$ so one sixth of the people live in California and five sixth Elsewhere. Note that we can also think of the equilibrium state P as $P_\infty = \lim_{k\to\infty} P_k = \lim_{k\to\infty} T^k P_0$.
Here we are assuming this limit exists (it does).

Example 2 There are two local branches of the Limousine Rental Company, one at the Airport and one in the City, as well as branches Elsewhere.

Say every week of the limousines rented from the Airport 25% are returned to the City and 2% to branches located Elsewhere. Similarly of the limousines rented from the City 25% are returned to Airport and 2% to Elsewhere. Finally, say 10% of the limousines rented from Elsewhere are returned to the Airport and 10% to the City.

If initially there are 35 limousines at the Airport, 35 in the City, and 150 Elsewhere, what is the long-term distribution of the limousines?

Let A_k be the percentage of limousines at the Airport at the beginning of week $k = 0, 1, \ldots$, C_k at the City, and E_k elsewhere. Then, as above,

$$\begin{aligned} A_{k+1} &= .73A_k + .25C_k + .1E_k \\ C_{k+1} &= .25A_k + .73C_k + .1E_k \\ E_{k+1} &= .02A_k + .02C_k + .8E_k \end{aligned}$$

We again write this in the matrix form $P_{k+1} = TP_k$, $k = 0, 1 \ldots$ and seek an equilibrium state: $P = TP$. It is $A = C = 5E$. Since $A + C + E = 1$ this gives $A = C = 5/11$ and $E = 1/11$. Initially there were $35 + 35 + 150 = 220$ limousines, so the equilibrium distribution is 100 at the Airport and City, while 20 elsewhere.