Math 210 — Exam 1 — Jerry L. Kazdan
March 12, 2019 — 1:30–2:50

DIRECTIONS: Part A has 4 short questions (10 points each), Part B has 3 traditional problems (20 points each),
To receive full credit your solution should be clear and correct. Neatness counts. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3 ×5 with notes on both sides.

PART A: Four shorter problems, 10 points each [total: 40 points]

A-1. Say you have k linear algebraic equations in n variables; in matrix form we write $AX = Y$. Give an *explicit* counterexample for each of the following assertions.

a) If $n = k$ there is always *at most one* solution.

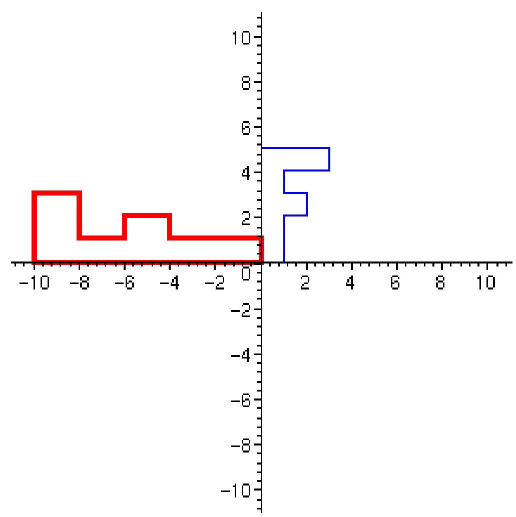
b) If $n > k$ you can *always* solve $AX = Y$.

c) If $n < k$ the *only* solution of $AX = 0$ is $X = 0$.

A-2. Let S and T be linear spaces and $L : S \to T$ be a linear map. Say v_1 and v_2 are (distinct!) solutions of the equations $Lx = y_1$ while w is a solution of $Lx = y_2$. Answer the following in terms of v_1, v_2, and w.

a) Find some solution of $Lx = 2y_1 - 3y_2$.

b) Find another solution (other than w) of $Lx = y_2$.



A-3. Find a 2×2 matrix A that in the standard basis transforms the larger **F** (on the left) to the smaller.

A-4. Let X and Y be vectors in $\mathbb{R}^n$. If the Pythagorean Theorem holds:

$$\|X + Y\|^2 = \|X\|^2 + \|Y\|^2,$$

show that X and Y are orthogonal.

Part B: Three traditional problems, 20 points each [60 points]

B-1. In a large city, a car rental company has three locations: the Airport, the City, and the Suburbs. One has data on which location the cars are returned daily:

- Rented at Airport: 5% are returned to the City and 20% to the Suburbs. The rest are returned to the Airport.
- Rented in City : 10% are returned to Airport, 10% returned to Suburbs. The rest are returned to the City.
- Rented in Suburbs: 20% are returned to the Airport and 5% to the City. The rest are returned to the Subirbs.

If initially there are 20 cars at the Airport, 65 in the city, and 15 in the suburbs, what is the long-term distribution of the cars?

B-2. A friend is tested for a relatively rare cancer that occurs in only 1 out of every 10,000 people his age. The test is accurate in the sense that:

- 10% of those who do not have the cancer still test positive (false positives)
- 2% of those who have the cancer test negative (false negatives).

If your friend tests positive, what is the likelihood that he has the cancer?

[*There is no need to do any arithmetic to "simplify" your result. I'll assume you can do that.*]

B-3. [Making a Triangle]

a) First a calculation. Choose independently two numbers x and y at random from the unit interval $0 \le t \le 1$. This defines a random point (x, y) in the unit square. Find the probability that this point is in the set

$$S := \{\ x \le y, \quad \text{and} \quad x \le 1/2, \quad \text{and} \quad y - x \le 1/2, \quad \text{and} \quad 1 - y \le 1/2\ \}.$$

b) You want to make a triangle whose sides have length a, b, and c with $a + b + c = 1$. Which numbers will work? If any side has length greater than 1/2 then the triangle can't close up. Thus we need

$$0\underbrace{\;\text{—}\;}_{a}\underbrace{\;\text{————}\;}_{b}\underbrace{\;\text{—}\;}_{c}1$$

$$a \le 1/2, \qquad b \le 1/2, \quad \text{and} \quad c \le 1/2, \quad \text{along with} \quad a + b + c = 1.$$

To determine the pieces a, b, and c, choose two numbers u and v at random from the unit interval $0 \le t \le 1$. There are two cases: if $u \le v$: $0\text{—}\overset{\text{u}}{\text{———}}\overset{\text{v}}{\text{—}}1$, then let $a = u$, $b = v - u$, and $c = 1 - v$. The other case, $v \le u$, we similarly let $a = v$, $b = u - v$, and $c = 1 - u$.

Now use part a) to determine the probability that the three pieces can be used to form a triangle.