Signature

PRINTED NAME

Math 210 — **Exam 1** — Jerry L. Kazdan

March 12, 2019 — 1:30–2:50

DIRECTIONS: Part A has 4 short questions (10 points each), Part B has 3 traditional problems (20 points each),
To receive full credit your solution should be clear and correct. Neatness counts. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3×5 with notes on both sides.

PART A: Four shorter problems, 10 points each [total: 40 points]

Score	
A-1	
A-2	
A-3	
A-4	
B-1	
B-2	
B-3	
Total	

A-1. Say you have k linear algebraic equations in n variables; in matrix form we write $AX=Y$. Give an *explicit* counterexample for each of the following assertions.

a) If $n=k$ there is always *at most one* solution.

b) If $n>k$ you can *always* solve $AX=Y$.

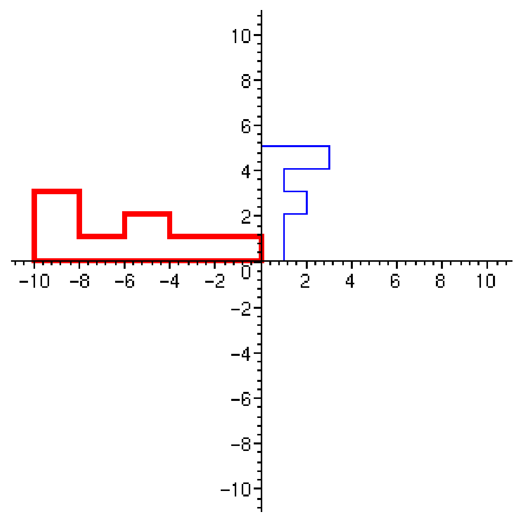
c) If $n<k$ the *only* solution of $AX=0$ is $X=0$.

A-2. Let S and T be linear spaces and $L : S \to T$ be a linear map. Say v_1 and v_2 are (distinct!) solutions of the equations $Lx = y_1$ while w is a solution of $Lx = y_2$. Answer the following in terms of v_1, v_2, and w.

a) Find some solution of $Lx = 2y_1 - 3y_2$.

b) Find another solution (other than w) of $Lx = y_2$.

A-3. Find a 2×2 matrix A that in the standard basis transforms the larger **F** (on the left) to the smaller.



A-4. Let X and Y be vectors in $\mathbb{R}^n$. If the Pythagorean Theorem holds:

$$\|X+Y\|^2 = \|X\|^2 + \|Y\|^2,$$

show that X and Y are orthogonal.

PART B: Three traditional problems, 20 points each [60 points]

B-1. In a large city, a car rental company has three locations: the Airport, the City, and the Suburbs. One has data on which location the cars are returned daily:

- RENTED AT AIRPORT: 5% are returned to the City and 20% to the Suburbs. The rest are returned to the Airport.
- RENTED IN CITY : 10% are returned to Airport, 10% returned to Suburbs, the rest to the City.
- RENTED IN SUBURBS: 20% are returned to the Airport and 5% to the City, the rest returned to the Suburbs.

If initially there are 20 cars at the Airport, 65 in the city, and 15 in the suburbs, what is the long-term distribution of the cars?

B-2. A friend is tested for a relatively rare cancer that occurs in only 1 out of every 10,000 people his age. The test is accurate in the sense that:

- 10% of those who do not have the cancer still test positive (false positives)
- 2% of those who have the cancer test negative (false negatives).

If your friend tests positive, what is the likelihood that he has the cancer?

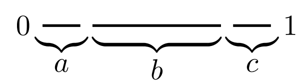
[*There is no need to do any arithmetic to "simplify" your result. I'll assume you can do that.*]

B-3. [Making a Triangle]

a) First a calculation. Choose independently two numbers x and y at random from the unit interval $0 \le t \le 1$. This defines a random point (x, y) in the unit square. Find the probability that this point is in the set

$$S = \{\ x \le y, \quad \text{and} \quad x \le 1/2, \quad \text{and} \quad y - x \le 1/2, \quad \text{and} \quad 1 - y \le 1/2\ \}.$$

b) You want to make a triangle whose sides have length a, b, and c with $a + b + c = 1$. Which numbers will work? If any side has length greater than $1/2$ then the triangle can't close up. Thus we need



$$a \le 1/2, \qquad b \le 1/2, \quad \text{and} \quad c \le 1/2, \quad \text{along with} \quad a + b + c = 1.$$

To determine the pieces a, b, and c, choose two numbers u and v at random from the unit interval $0 \le t \le 1$. There are two cases: if $u \le v$: 0 —u————v— 1, then let $a = u$, $b = v - u$, and $c = 1 - v$. The other case, $v \le u$, we similarly let $a = v$, $b = u - v$, and $c = 1 - u$.

Now use part a) to determine the probability that the three pieces can be used to form a triangle.

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