CONNECTING AND RESOLVING SEN'S AND ARROW'S THEOREMS

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Abstract. It is shown that the source of Sen's and Arrow's impossibility theorems is that Sen's Liberal condition and Arrow's IIA counter the critical assumption that voters' have transitive preferences. As this means the admissible procedures cannot distinguish whether voters have transitive or irrational cyclic preferences, the Pareto condition forces cycles. Once this common cause of these perplexing conclusions is understood, these classical conclusions end up admitting quite benign interpretations where it becomes possible to propose several resolutions.

After several decades Arrow's (1952) Impossibility Theorem and Sen's (1970) "The impossibility of a paretian liberal" retain their central positions in the large literature they have spawned. On the surface, these results are quite different. Sen, for instance, comments that "unlike in the theorem of Arrow, we have not required transitivity of social preference. We have required . . . merely the existence of a best alternative in each choice situation." Sen further notes that he has "not imposed Arrow's much debated condition of 'the independence of irrelevant alternatives.' " Nevertheless, as demonstrated here, Sen's and Arrow's Theorems share the same (surprisingly elementary) explanation.

By exposing the common cause for these assertions, both conclusions become obvious and resolutions become easy to offer. To illustrate, instead of the traditional dire and almost Draconian interpretations usually assigned to Arrow's theorem – the sense that his theorem requires us to suffer either a dictator or a paradox – we find that his theorem reduces to the benign and innocuous conclusion that "if a group wants rational outcomes, they cannot use procedures intended for unsophisticated, irrational voters." In turn, this (accurately) suggests that resolutions will reflect the sense that "if a group wants rational outcomes, they should use procedures designed for rational voters." It is interesting that the argument developed here also explains related problems from other areas; e.g., consumer surplus, etc.

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1. Sen's Theorem

In his theory, which is of interest only for $k \geq 3$ alternatives, Sen uses the axioms

(U) (*Unrestricted Domain.* Every logically possible set of individual orders is included in the domain of the collective choice rule.)

(P) (*Pareto.* If every individual prefers any alternative a to another alternative b, then society must prefer a to b.)

(ML) (*Minimal Liberalism.* There are at least two individuals where each is decisive over at least one pair of alternatives; i.e., there is a pair $\{a, b\}$ such that if a decisive individual over this pair prefers $a \succ b$ (respectively $b \succ a$), then society prefers $a \succ b$ (respectively $b \succ a$).

Sen proves that all procedures satisfying these axioms admit cycles. Cycles are undesirable, of course, because they frustrate the identification of a maximal element. An important assumption for his theorem is that the preferences of the $n \geq 2$ voters are transitive. After all, Sen's conclusion would have no interest should voters have cyclic preferences because (P) would mandate a cyclic outcome for the unanimity profile where everyone has the cyclic preferences $a \succ b$, $b \succ c$, $c \succ a$. This cycle is not disturbing nor even very interesting because it only manifests the "garbage in, garbage out" adage that if structure is not imposed on inputs, then we cannot expect structure on outputs. It is to avoid this "garbage in" effect that transitivity of preferences is a critical assumption for choice theory.

The surprise is that, contrary to expectations and intentions, *all procedures satisfying ML abolish this transitivity assumption. As I show, ML procedures are specifically required to service voters so unsophisticated that they can have these primitive cyclic and other nontransitive preferences.* Indeed, as I prove, if a procedures can serve only voters with transitive preferences, such as the standard plurality vote, it is immediately eliminated by ML. So, by understanding that ML dismisses the critical assumption about the transitivity of voters, rather than being worrisome or paradoxical, Sen's conclusion must be expected. The real mystery would be if his conclusion were otherwise.

To explain, because axioms *exclude* procedures, a way to understand a theorem is to identify which methods are dismissed by each axiom. For instance, (P) dismisses all positional methods (i.e., procedures where points are assigned to candidates according to how a voter ranks them) which award a single point to each of a voter's j top-ranked candidates and zero to all others. So, $j = 1$ is the plurality vote, $j = k-1$ is the antiplurality vote. The plurality vote is excluded, for example, because its $a \succ b \sim c$ ranking for the unanimity profile $a \succ b \succ c$ fails to respond to the unanimous $b \succ c$ preference.

Rather than worrying about (P), our main concern is to appreciate the kind of procedures dismissed by ML. To do so, consider Sen's proof for the $k = 3$ alternatives $\{a, b, c\}$ where voters one and two determine, respectively, the $\{a, b\}$ and the $\{a, c\}$ outcomes. As only the decisive voter determines the outcome of a designated pair, restrictions cannot be imposed on the other voters' rankings for the pairs labeled "none" in Table 1. Consequently, a ML procedure can neither recognize nor use these binary rankings; it cannot even check whether they support or deny transitivity.

(1)

Voter	$\{a,b\}$	$\{b,c\}$	$\{a,c\}$
1	–	–	none
2	none	–	–
Others	none	–	none

Instead of embracing the critical assumption of transitivity, the true domain of a ML admissible procedure (for $k=3$ and where the decisive voters have power over the indicated pairs) is the set of all profiles where the "none" slots can be filled in any desired manner. Denote this domain by $SEN^n(3)$. Notice that $SEN^n(3)$ only requires a voter to rank each pair; the pairwise rankings need not satisfy *any* sequencing requirement such as transitivity. It is equally obvious that all ML procedures are defined on $SEN^n(3)$. (This automatically eliminates all positional voting methods because they require at least acyclic preferences.) Thus, it is appropriate to treat the ML admissible procedures as being intended for societies so unsophisticated and primitive that the voters can only rank pairs.

While Sen obviously did not anticipate nor intend to use $SEN^n(3)$, it is a mandatory domain for ML procedures. The assumption of transitive preferences and U, then, constitute a *profile restriction* to $T^n(3) = \{$all n-person profiles where each person has transitive preferences of the three candidates and where there are no pairwise ties$\}$. To connect this profile restriction with standard choice issues, recall that even though (by assumption) voters have transitive preferences, we still incur cyclic pairwise election outcomes. One way to remove these cycles is to impose sufficiently severe profile restrictions, such as Black's single-peakedness (Black 1958, Saari 1994, 1995b), to ensure acyclic outcomes. (Later I indicate why these restrictions work.) Similarly, in our process of finding a new interpretation for Sen's Theorem by examining which procedures are removed by each condition, the next step is to retain only those ML procedures which adequately serve the sophisticated voters modelled by preferences in $T^n(3)$; Sen's definition of "adequate" requires acyclic outcomes.

A way to identify these ML procedures is to find those methods capable of distinguishing between transitive and nontransitive preferences. That is, we need to determine whether any ML procedure can detect differences between $T^n(3)$ and $SEN^n(3)$. To see what is involved by using Sen's example (but with a significantly different interpretation), the following table displays the portions of his profile recognized by a ML procedure. If a ML procedure can distinguish between $T^n(3)$ and $SEN^n(3)\backslash T^n(3)$, then either (1) it is impossible to fill the blanks to create transitive rankings (so this partial profile cannot occur with the $T^n(3)$ restriction), or (2) the only ways to fill the blanks result in transitive preferences (so some procedure may recognize transitive preferences).

(2)

1	$a \succ b$	$b \succ c$	–
2	–	$b \succ c$	$c \succ a$
Others	–	$b \succ c$	–

As it is trivial to fill the blanks to violate transitivity (so 2 fails), it remains (1) to determine whether they can be filled with transitive rankings. They can; for voter-one, use $a \succ c$, for voter-two use $b \succ a$, and for all others use $a \succ b$, $a \succ c$. Thus the $T^n(3)$ profile restriction fails because *a ML procedure cannot distinguish whether it is using transitive preferences from $T^n(3)$ or nontransitive $SEN^n(3)$ preferences;* it cannot discriminate between rationality and unsophisticated voters with primitive cyclic preferences. The transitivity dismissed by ML remains lost.

To make this claim more precise, by modifying the above argument it is easy to prove the following:

Proposition 1. *Let $\mathbf{p} \in SEN^n(3)$. There exists a profile $\mathbf{p}_t \in T^n(3)$ so that a ML procedure cannot distinguish between $\mathbf{p}$ and $\mathbf{p}_t$. Conversely, for $\mathbf{p}_t \in T^n(3)$, not only are there profiles $\mathbf{p}_c \in SEN^n(3) \backslash T^n(3)$ that a ML procedure cannot differentiate from $\mathbf{p}_t$, but $\mathbf{p}_c$ can be chosen so that all voters have cyclic preferences.*[1]

According to Prop. 1, the transitivity restriction has *not*, in any manner, changed or restricted the perceivable domain for a ML procedure; instead, all ML procedures must behave as though the domain remains $SEN^n(3)$. This is because the portion of a $SEN^n(3)$ profile recognized by a ML procedure also occurs for a profile in $T^n(3)$. Indeed, a ML procedure cannot even determine whether a voter is transitive or cyclic! Thus all ML outcomes that arise by using the primitive voter domain $SEN^n(3)$ also must occur with transitive voters! The ML assumption forces its admissible procedures to be so dedicated toward primitive preferences that the profile restriction makes no difference!

This proposition makes it trivial to construct examples of transitive profiles forcing cyclic outcomes. Just start with a profile $\mathbf{p}_c \in SEN^n(3)$ where a cycle is the natural outcome (because of (P) or some other condition such as maximin principle used in (Gaertner, Pattanaik, and Suzumura, 1992)), and then construct one of the guaranteed (Prop. 1) transitive profiles $\mathbf{p}_t \in T^n(3)$ that is ML indistinguishable from $\mathbf{p}_c$. A natural $\mathbf{p}_c$ candidate is the unanimity cyclic profile $a \succ b$, $b \succ c$, $c \succ a$ because (P) forces the $\mathbf{p}_c$ outcome to be the same cycle. A choice for the guaranteed indistinguishable $\mathbf{p}_t$ is the transitive profile constructed for Table 2. Thus the only fair outcome (i.e., the outcome demanded by (P)) for a profile with the partial listing of Table 2 is a cycle. From this perspective, rather than being a surprise, cyclic outcomes must be expected. Transitivity remains a ML hostage that (because of Prop. 1) cannot be easily released.

The same argument holds for $k \geq 4$ where voters one and two determine respectively, the $\{a, b\}$ and the $\{c, d\}$ rankings. As a ML procedure cannot recognize the binary preferences of nondecisive voters for these pairs, the ML effective domain imposes no restrictions on how a nondecisive voter ranks them. Again, counter to expectations and intentions, ML destroys the assumption of transitive preferences. Instead, a procedure satisfying ML is defined over a larger domain[2] $SEN^n(k)$ where

[1] So, two profiles are equivalent if they agree on all pairwise rankings except those noted as "none" in Table 1; the equivalence classes defined by this relationship partition $SEN^n(3)$. Clearly, a ML procedure cannot distinguish between profiles in the same equivalence class. The proposition states that each equivalence class has at least one profile from $T^n(3)$ as well as at least one profile from $SEN^n(3) \backslash T^n(3)$ where each voter is cyclic. As $T^n(3)$ does not reduce the number of $SEN^n(3)$ equivalence classes, each transitive profile can be misinterpreted as being a cyclic profile.

[2] To define $SEN^n(k)$, start with transitive preferences, and vary the rankings of the indicated

all procedures intended for rational voters are dismissed, where most preferences fail transitivity, and (by P) where many ML outcomes are cyclic. Again, the obvious and immediate extension of Prop. 1 asserts that for every $\mathbf{p} \in SEN^n(k)$, there is a ML indistinguishable transitive profile $\mathbf{p}_t \in T^n(k)$. Consequently the profile restriction $T^n(k)$ does not, in any way, alter or restrict the domain for a ML procedure from that of $SEN^n(k)$.

Again, cyclic outcomes are trivial to generate by choosing a $\mathbf{p}_c \in SEN^n(k)$ where the only (P) fair outcome is a cycle and then finding an indistinguishable transitive profile. One $\mathbf{p}_c$ choice assigns all voters the cyclic rankings $a \succ b, b \succ c, c \succ d, d \succ a$ so (by P) the cycle is the only fair outcome. To find an indistinguishable $\mathbf{p}_t \in T^n(k)$, list the $\mathbf{p}_c$ portions that a ML procedure can examine.

(3)

Voter	$\{a,b\}$	$\{b,c\}$	$\{c,d\}$	$\{a,d\}$
1	$a \succ b$	$b \succ c$	–	$d \succ a$
2	–	$b \succ c$	$c \succ d$	$d \succ a$
Others	–	$b \succ c$	–	$d \succ a$

Generating a ML equivalent, transitive profile $\mathbf{p}_t$ is immediate; e.g., let voter-one have the preferences $d \succ a \succ b \succ c$ and all other voters have $b \succ c \succ d \succ a$.

Salles (1994) escalates the complexity of the issue by creating a troubling situation where the same transitive profile generates *two* cycles. Using our notation, his Mr. Prude and Mr. Lascivious (an extension of one of Sen's examples) have, respectively, the preferences $a \succ b \succ c \succ d$ and $d \succ b \succ c \succ a$. Salles' description of these alternatives permits him to appeal to Hammond's (1982) condition about privately unconditional preferences so that P is decisive over $\{a, b\}$, $\{c, d\}$ while L is decisive over $\{a, c\}$, $\{b, d\}$. This combination of decisiveness and specified preferences defines

(4)

P	$a \succ b$	–	$c \succ d$	–	$b \succ c$
L	–	$c \succ a$	–	$d \succ b$	$b \succ c$

To understand Salles' cycles note that Table 4 also admits the indistinguishable nontransitive unanimity profile $a \succ b$, $c \succ a$, $d \succ b$, $c \succ d$, $b \succ c$ which requires the outcome to include the two (P) cycles $a \succ b \succ c \succ a$ and $b \succ c \succ d \succ b$. Again, problems are encountered because the critical transitivity assumption is lost when "natural conditions" force the domain – and the admissible procedures – to admit indistinguishable nontransitive voters. Similarly, now that we understand the basic principle behind the Sen-type conclusions, it is easy to design more complex examples.

So, rather than being a mystery or surprise, Sen's Theorem is to be expected if only because ML procedures fail to distinguish between $T^n(k)$ and $SEN(k)$. From this perspective, Sen's Theorem only asserts that a procedure intended for primitive, unsophisticated preferences cannot meet the needs of sophisticated (transitive)

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pairs in all possible ways. This process defines equivalence classes of preferences (i.e., sets of transitive and nontransitive profiles that cannot be distinguished), so the extension of Prop. 1 is immediate. However, $SEN(k)$ for $k \geq 4$ can require "partial transitivity" for triplets of pairs not assigned to decisive agents. For instance, in Table 3, voter-one's missing rankings must be $a \succ c$, $d \succ b$.

voters; transitivity is much too weak of a profile restriction. While we could try to impose effective profile restrictions[3], stronger restrictions do not alter the fact that we still are dealing with crude procedures defined for the nontransitive preferences of $SEN^n(k)$. Consequently, even with severe profile restrictions, stilted conclusions and procedures must be expected. A better approach is to replace ML and (P) with more reasonable conditions which admit a domain which better approximates properties of transitive voters; this is discussed in Section 5.

2. A mathematical explanation

Once $k \geq 4$, it turns out that the procedures admitted by Arrow's Independence of Irrelevant Alternative (IIA) are directed toward voters with preferences even more primitive than those introduced by Sen's Theorem! This is because $SEN^n(k)$ can require partial transitivity, but (as we will see) the IIA admissible procedures are forbidden from recognizing even the slightest hint of rationality. Before providing details, it is worth anticipating objections (and addressing concerns raised by readers of earlier drafts of this paper) about the new, unorthodox explanation offered here.

To start, notice that the strength of Arrow's and Sen's Theorems is that they are mathematical conclusions. Both theorems have withstood almost a half century of attacks because they are based on mathematic logic and analysis. Therefore, any criticism and any defense must follow these rules.

What are the mathematical rules? The first is that no matter how much we may wish to do so, *we cannot define the natural (maximal) domain for a class of procedures*; instead, the appropriate domain is defined by the specified mathematical properties. For instance, no matter how hard I may wish to include the point $x = 4$ in the domain for $f(x) = \frac{1}{x-4}$, mathematics dictates that the correct domain is a subset of $(-\infty, 4) \cup (4, \infty)$. Similarly, no matter how much we may want the domain for Sen's ML to be the transitive preferences, the appropriate domain dictated by ML is $SEN^n(k)$. Thus, the mathematical properties of the assumed or imposed conditions for the procedures define the appropriate domain; if we try to impose the domain without regard to these properties, then we must expect the penalty of impossibility assertions.

To illustrate with an elementary example, consider the following axiom:

Axiom 1. *For real numbers a and b, there is a real number c satisfying the property that $b + c = a$.*

This axiom only defines subtraction; $c = a - b$. Now suppose we impose the domain $R_+ = (0, \infty)$ of positive numbers. It is obvious that the incompatibility of the imposed domain R_+ for a, b, c and Axiom 1 cause an *impossibility assertion.* (Once $b > a > 0$, no $c \in R_+$ exists.) It is equally obvious that this impossibility theorem is of no surprise, interest, nor concern; it only means that R_+ is an inappropriate domain for the axiom. To impose R_+ without examining whether it is compatible with the axiom is mathematically naive.

[3]An effective profile restriction eliminates those *equivalence classes* that have a $SEN^n(k)$ profile where P requires the ML procedures to have cyclic conclusions; the restriction need not be stated in terms of transitive preferences!

The same notion holds throughout mathematics. For instance, for a set of elements, G, that are combined with an operation $\circ$, suppose the goal is to solve all problems of the form $a \circ x = b$, $y \circ a = b$ for any a, $b \in G$. The well-known resolution requires G to define a correct domain relative to $\circ$; namely, $< G, \circ >$ must satisfy the properties of a *group*. Conversely, if $< G, \circ >$ does not define a group, then we will suffer an impossibility theorem. But, this impossibility assertion is not of surprise nor interest; it just means that we are not using the domain required by the desired properties.

The same analysis holds for Arrow's and Sen's Theorems. Rather than imposing the domain by fiat or assumption, the appropriate domains are defined by the specified properties of ML and IIA. So, while it may seem to make excellent sense to assume from the very beginning – before introducing an axiom of any substance – that the preferences are transitive, we now see that such an approach is mathematically naive. Indeed, if such an approach were allowed, any number of impossibility assertions could be published; e.g., we could assume – before considering Axiom 1 – that the correct domain of interest is R_+. By imposing the domain, rather than finding it from the basic assumed properties, impossibility conclusions become trivial to derive. Stated in another fashion, impossibility assertions only manifest that the axioms are incompatible; by computing the natural domain, we can understand where the incompatibility arises.

Once a domain is defined, it is the domain – not us – which determines the admissible procedures. For instance, the specified domain $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$ does not admit the function $f(x) = \frac{1}{x-4}$. Similarly, by determining the domains required by ML and IIA, we discover that *all procedures which can be used only by rational voters (such as the Borda Count) or even only by acyclic voters (such as the plurality vote) are immediately excluded!* (This argument is made precise in the next section.) Instead, ML and IIA limit us to procedures so crude that they can be viewed as being designed for unsophisticated irrational voters. But, by being required to use only highly unsophisticated procedures, it is not surprising that sophisticated outcomes (i.e., transitive rankings) need not occur. This new, intuitive argument accurately captures the essence of Sen's and Arrow's Theorems.

Of course, one might argue that the real goal is to understand "the level of sensitivity of these crude procedures" when they are restricted to transitive preferences. From the mathematics, this means we need to understand whether the admissible procedures can detect any difference in the irrational or rational domains. To see the mathematical notions that are involved with this profile restriction, let $z = f(x, y) = \frac{1}{x-4}$. To avoid the pathology caused by the point $x = 4$, we might restrict attention to the line $y = 10$. The reason this is a useless restriction is that the $y = 10$ line meets every f level set. Thus, all f behavior – including that near the point $x = 4$ – still occurs. As demonstrated in the last section, an identical effect occurs with Sen's Theorem; the restriction to transitive preferences resembles the $y = 10$ restriction because the set of transitive preferences meets all $SEN^n(k)$ level sets. This means that the transitivity restriction is useless; it has absolutely no impact on eliminating the pathologies caused by the $SEN^n(k)$ irrational preferences. In turn, this means that the ML and IIA procedures exhibit absolutely no sensitivity to whether voters are irrational or rational!

This sensitivity issue can be further illustrated with Axiom 1. We may wish to determine how sensitive procedures satisfying Axiom 1 are to the domain restriction R_+ – none are. But, this is no surprise once we recognize what is the correct domain required by the axiom. A similar explanation holds for Arrow's and Sen's assertions.

So, while my approach is unorthodox because it never has been used before in choice theory, it just follows standard mathematical analysis. Conversely, the orthodox, traditional approach ignores domain considerations with the cost of obscuring for nearly a half century the surprisingly simple explanations for Sen's and Arrow's assertions.

3. Arrow's Theorem

To explain Arrow's Theorem for $k \geq 3$ alternatives, recall his axioms

(IIA) (*Independence of irrelevant alternatives.* The relative group ranking of any two candidates only depends upon the voters' relative ranking of this pair.)

(ND) (*No dictator.* The group outcome cannot always be the same as the ranking of a particular voter.)

Arrow concludes that if the voters' preferences are transitive and if the outcomes must be transitive, then the only procedure satisfying U, P, IIA is a dictator; namely, U, P, IIA, and ND are in conflict. The real problem is to understand why.

As true with Sen's Theorem, Arrow's conclusion would have no interest should voters have cyclic preferences (because (P) would require cyclic outcomes). Nevertheless, against expectations and intentions, the true domain defined by IIA welcomes all possible kinds of unsophisticated voters with cyclic and other nontransitive types of preferences! Once the appropriate domain is determined, we discover the unexpected consequence that IIA retains only procedures intended for these highly unsophisticated voters with cyclic and primitive preferences! Moreover, as shown below, these IIA procedures are so crude, so insensitive to the notion that voters can have rational beliefs, that (with the exception of a dictator) they cannot distinguish between rational and highly irrational voters! Because IIA (as true for ML) dismisses the critical assumption of transitive preferences and admits only highly primitive procedures, rather than being surprising or disturbing, Arrow's conclusion must be anticipated.

To start with notation, if the k candidates are $\{c_1, c_2, \dots, c_k\}$ and $B(c_i, c_j) = \{c_i \succ c_j,\ c_j \succ c_i\}$, then $B(k) = \prod_{i<j} B(c_i, c_j)$ lists all $2^{\binom{k}{2}}$ ways to (strictly) rank each pair of alternatives. The product of $B(c_i, c_j)$ over all n voters, $B^n(c_i, c_j)$, lists all ways the voters can rank this pair while $B^n(k) = \prod_{i<j} B^n(c_i, c_j) = B(k) \times \cdots \times B(k)$ is the space of profiles where the only requirement is that each voter can rank each pair. As the pairwise rankings defining preferences in $B^n(k)$ need not be connected or related in any manner, they need not be transitive, quasitransitive, acyclic, or anything else. It is immediate that an IIA procedure must be defined on $B^n(k)$.

Actually, IIA imposes a more severe restriction. We see this from the usual IIA formulation requiring for any two profiles $\mathbf{p}^1$, $\mathbf{p}^2$, and any two social states c_i, c_j that if $\mathbf{p}^1_s$, $\mathbf{p}^2_s$ coincide on $\{c_i,\ c_j\}$ for all voters $s = 1, \dots, n$, then the outcomes of $\mathbf{p}^1$, $\mathbf{p}^2$ coincide on $\{c_i,\ c_j\}$. This condition requires for each pair that only the information about voters' preferences for this particular pair is admitted; all other information from each profile is totally ignored. So, for a given procedure

F, define $F_{c_i,c_j}(\mathbf{p})$ to be the $\{c_i, c_j\}$ relative rankings in $F(\mathbf{p})$. (To illustrate, if $F(\mathbf{p}) = c_3 \succ c_2 \succ c_4 \succ c_1$, then $F_{c_1,c_2}(\mathbf{p}) = c_2 \succ c_1$ and $F_{c_3,c_4}(\mathbf{p}) = c_3 \succ c_4$.) A procedure F satisfies IIA if and only if F admits this $\{F_{c_i,c_j}\}$ decomposition where the domain for each F_{c_i,c_j} is $B^n(c_i, c_j)$. Consequently, it follows immediately from this domain that IIA only requires the voters to rank each pair of alternatives; transitivity of personal preferences is a separate assumption.

Because IIA (as true with ML) makes F insensitive to certain aspects of the profile, a way to analyze Arrow's Theorem is to understand this ignored information.[4] This issue is closely related to the discussion of Sect. 2; namely, the domain defined by IIA determines the admissible procedures. By characterizing all procedures, we can discover the devalued kind of information. I show that IIA requires its admissible procedures to be totally insensitive to the rationality of voters! Arguably, this is a disqualifying characteristic.

To explain, notice that IIA eliminates any procedure that requires the voters to sequence their pairwise rankings in *any* manner. This is because each F_{c_i,c_j} outcome is totally independent of how each voter ranks any other pair. To illustrate, the plurality vote only requires voters to have a maximal candidate. Thus the plurality vote can be used by a voter with the cyclic preferences $c_2 \succ c_3$, $c_3 \succ c_4$, $c_4 \succ c_2$ as long as $c_1 \succ c_j$, $j = 2, 3, 4$. This partial sequencing suffices to cast this voter's plurality vote; one point is assigned to c_1 and zero to all others. However, IIA explicitly forbids using even this minimal sequencing information; it requires this voter's F_{c_1,c_2} vote to depend only the $c_1 \succ c_2$ ranking. Because an IIA procedure must ignore sequencing information, it cannot distinguish this setting from where c_3 is the voter's maximal candidate so c_1 and c_2 should receive zero points. Thus, the plurality vote fails IIA because it is sensitive to the partial rationality of voters – an IIA procedures cannot be.

Similarly, the only way ballots can be tallied with the Borda Count (BC) is if the voters have transitive rankings. (For k candidates, the BC assigns $k - j$ points to a voter's jth ranked candidate; $j = 1, \dots, k$.) As such, the BC is not an IIA procedure. If it were, then we would know how to tally a voter's ballot for c_2 and c_3 with just the $c_3 \succ c_2$ information. This is, of course, impossible because we do not know whether c_3 is top-ranked or ranked next to the bottom! In fact, in determining the $\{c_2, c_3\}$ ranking, IIA explicitly prohibits using any sequencing information about how c_2 or c_3 fare against other candidates. A similar argument shows that because all positional methods must be sensitive to whether voters are rational (or capable of some level of sequencing pairwise rankings), they are excluded by IIA.

Indeed, all procedures that can be used only by rational voters (as opposed to the unsophisticated, primitive voters capable only of ranking pairs) are immediately excluded by IIA. If a procedure can service only transitive voters, then the fact that a voter has the rankings $c_1 \succ c_2$ and $c_2 \succ c_3$ immediately imposes a condition on how the procedure handles this voter's $\{c_1, c_3\}$ ranking. But, this ability to forecast, to monitor the ability of voters to sequence pairwise rankings, is explicitly forbidden by IIA! In fact, this simple argument shows that *any procedure that can be used*

[4] Arrow and Sen knew that their conditions made the admissible procedures insensitive to certain kinds of information, but they did not identify the critical nature of the ignored information. I show it is the rationality of the voters.

only by voters capable of some level of sequencing pairwise rankings – no matter how crude – is immediately excluded by IIA. (These assertions follow immediately from the fact that $B^n(c_i, c_j)$ is the domain for F_{c_i,c_j}.) Because IIA procedures are insensitive to any ability to sequence pairwise rankings or to any level of rationality, we must view IIA methods as being designed for voters so unsophisticated and so primitive that they can only rank each pair; sequencing of the ranked pairs is beyond their talents.

Because IIA admissible procedures are totally insensitive to any rationality in the pairwise rankings, the IIA procedures cannot impose sequencing requirements on their pairwise outcomes. Consequently, transitive outcomes can be expected only with highly restrictive procedures in special settings (i.e., dictators with transitive preferences), or not at all. This provides a different, highly intuitive, and accurate interpretation of Arrow's Theorem.

To make this argument precise, the next step, as with Sen's Theorem, is to use (U) to impose $T^n(k)$ *as a profile restriction.* We need to determine whether this further reduces the class of IIA procedures. Namely, we want to find whether any mapping intended for the crude $B^n(k)$ preferences can recognize when it is dealing with rational voters (at least so that only transitive outcomes occur). A dictator is one such procedure; when restricted to transitive preferences, the dictator's binary rankings can be assembled in only one manner.

Sen's framework identifies the decisive agents so the same equivalence classes of profiles hold for all ML methods (Prop. 1). In Arrow's formulation we need to consider a procedure's "level sets" (of $B^n(k)$ profiles) defined for each outcome.[5] (By definition, a procedure is incapable of distinguishing between different profiles from the same level set.) Thus the argument remains essentially the same; because $B^n(k)$, $n \geq 2$, $k \geq 3$, is so huge, a level set of profiles always can be found which contains both a $\mathbf{p}_t \in T^n(k)$ and a $\mathbf{p}_c \in B^n(k) \backslash T^n(k)$ where the nontransitive profile $\mathbf{p}_c$ determines the procedure's "fair" but nontransitive outcome. Again, this reinforces the observation that IIA admissible procedures must ignore rationality!

It is worth illustrating this central assertion with the familiar pairwise vote (which is defined over $B^n(3)$). To do so, I need to find a transitive $\mathbf{p}_t$ and a non-transitive $\mathbf{p}_c$ which the pairwise vote finds indistinguishable and where the "fair" nontransitive outcome is determined by $\mathbf{p}_c$. So, decompose the Condorcet profile into its binary parts as given in the following table (Saari 1994, 1995b).

(5)

$a \succ b \succ c$	$(a \succ b)_1$	$(b \succ c)_2$	$(a \succ c)_3$
$b \succ c \succ a$	$(b \succ a)_3$	$(b \succ c)_1$	$(c \succ a)_2$
$c \succ a \succ b$	$(a \succ b)_2$	$(c \succ b)_3$	$(c \succ a)_1$

By satisfying anonymity, the pairwise vote cannot determine who cast what ballot, so it cannot distinguish the Condorcet profile from the profile $\mathbf{p}_c$ where the pairwise rankings of voters 1, 2, 3 are specified by the subscripts. Thus, the pairwise vote (or any procedure satisfying IIA and anonymity) cannot distinguish between the

[5] An assertion such as Prop. 1 does not hold in Arrow's setting. For instance, with the pairwise vote, there is no transitive profile that is indistinguishable from the unanimous cyclic profile $\mathbf{p}_c \in B^n(3)$ where $a \succ b$, $b \succ c$, $c \succ a$.

Condorcet profile and the profile of cyclic, confused voters where voters one and two have the cyclic preferences $\{a \succ b, b \succ c, c \succ a\}$ and voter three has the reversed cyclic preferences $\{a \succ c, c \succ b, b \succ a\}$. This confused profile has two voters in favor of a proposition (the list of cyclic rankings) and one directly against, so the fair outcome is to adopt the proposition with its 2:1 vote. But because the Condorcet and the confused profiles are indistinguishable, both profiles must have the same cyclic pairwise outcome. Thus *the cycle is the only fair pairwise outcome for a profile with the binary components of Table 5.*

In particular, this example (accurately) suggests that the true interpretation of the widely studied pairwise voting cycles is just that the pairwise vote procedure ignores the transitivity of preferences. Notice how this comment seriously impugns the integrity of methods based on pairwise rankings; this includes the widely accepted standard of the "Condorcet winner." (For different arguments, see (Saari, 1995b).) Equally important, it explains the seeming conflict that Arrow's Theorem requires us to choose between a dictator or a paradox. Once we learn that we cannot trust totally separated pairwise comparisons (because they lose the assumption of transitive preferences) the problem disappears – we have to ignore the pairwise information (because *they* may create the paradox) and trust the ranking of all candidates.

The Table 5 decomposition applies for IIA procedures satisfying anonymity, but it is not applicable for a procedure where individuals have varying levels of influence on the outcome. To handle all possible situations, think of an IIA procedure as defining a sense of "fairness" as manifested by which of the voters' binary rankings determine each relative ranking of each pair. As I now show, whatever this sense of fairness, each ND, IIA procedure defines equivalence classes of profiles (i.e., profiles among which the procedure cannot differentiate) with transitive and nontransitive profiles but where some of the nontransitive profiles dictate a nontransitive outcome based on the procedure's "fairness" criteria. Thus the argument used with the pairwise vote extends to all IIA procedures.

This goal is accomplished by using the proof of (an extended version of) Arrow's Theorem in (Saari, 1994, 1995b). This proof reduces to examining the properties of a partial profile where two voters are decisive over different pairs, say $\{a, b\}$ and $\{b, c\}$. A voter, however, may be decisive over an assigned pair only in *certain specified situations* (i.e., all other voters may need to have specified rankings of the pair). The argument used to generate a cycle requires voter-one to change $\{a, b\}$ rankings while keeping fixed both a specified $\{b, c\}$ ranking and an unspecified $\{a, c\}$ ranking. Similarly, voter-two has to vary $\{b, c\}$ rankings while keeping fixed both a specified $\{a, b\}$ and an unspecified $\{a, c\}$ ranking.[6] If a ND, IIA procedure can recognize when the voters have transitive preferences, then either it is impossible to fill in the "fixed" regions with binary rankings that are transitive (so this situation cannot occur with the $T^n(k)$ profile restriction), or these regions can be filled only

[6]This suffices because after the fixed $\{a, c\}$ ranking is determined, say, $c \succ a$, then voters one and two can choose preferences so that the group's rankings are $a \succ b$ and $b \succ c$.

in a transitive manner.

(6)

Voter	$\{a, b\}$	$\{b, c\}$	$\{a, c\}$
1	Varies	Specified	Fixed
2	Specified	Varies	Fixed
Others	Specified	Specified	Fixed

Without transitivity, the "fixed" blanks can be filled in any desired way, so it remains to determine whether they can be filled while preserving transitivity. They can; vary the first voter's rankings between $a \succ b \succ c$, $b \succ a \succ c$ or between $c \succ a \succ b$, $c \succ b \succ a$ where the choice depends upon the specified $\{b, c\}$ ranking. (In either case the $\{a, c\}$ and $\{b, c\}$ rankings remain fixed.) Similarly, the second voter varies between $a \succ b \succ c$, $a \succ c \succ b$ or $b \succ c \succ a$, $c \succ b \succ a$ depending on the required $\{a, b\}$ ranking. For each of the remaining voters, once the $\{a, b\}$ and $\{b, c\}$ ranking is specified, an $\{a, c\}$ choice can be made to ensure transitivity.

Because IIA prohibits imposing any requirement upon the $\{a, c\}$ ranking, other than it remains fixed, the procedure cannot distinguish between certain transitive and nontransitive profiles. Thus cycles can be viewed as manifesting a "fairness" property of the IIA procedure which is dictated by a nontransitive profile; the partial portion of the nontransitive profile is indistinguishable from the partial profile of transitive preferences. In particular, the $T^n(k)$ restriction does not allow ND, IIA respecting procedures to recognize transitive preferences. As this effectively returns us to the realm of nontransitive voters, Arrow's conclusion is to be expected.

This argument is essentially the same as our argument explaining Sen's Theorem. This is no coincidence because both theorems revolve about conditions which make rational and irrational beliefs indistinguishable. In both cases, the tables are convenient devises to describe the true domains of the admissible procedures.

4. IIA TYPE CONDITIONS

Similar assertions, supported by the same arguments, must be anticipated whenever the rankings of subsets of alternatives are based only on portions of voters' preferences. By ignoring viable substitutes, a procedure's true domain involves equivalence classes of preferences where most of them fail transitivity. Transitive preferences may be intended, but the domain may require transitivity only over subsets of alternatives. So, by using the "mix-and-match" approach of Table 5 with sufficiently heterogeneous preferences and enough voters (or by using the argument applied to Arrow's assertion), settings can be found where it is impossible for procedures to distinguish between transitive and intransitive preferences! This means that until the spirit of IIA is replaced with conditions which allow different subsets of substitutes to be related, we must expect that the critical assumption of transitive preferences is useless. IIA and related conditions determine the true domain of procedures; our best wishes or intentions do not.

While these comments can be illustrated with consumer surplus, price models, etc., I stay in the category of choice theory and illustrate it with $k = 4$ candidates. Consider an IIA condition where the relative ranking of any triplet depends only on the voters' relative rankings of this triplet. The effective domain requires each voter to have a transitive ranking for each triplet but, because IIA decouples the subsets, these rankings need not be related in any manner. Thus the true domain required by this IIA assumption admits, for instance, a voter with the cyclic preferences (constructed with triplets) $a \succ b \succ c$, $b \succ c \succ d$, $c \succ d \succ a$, $d \succ a \succ b$. Again, the profile restriction $T^n(k)$ need not permit admissible procedures to recognize rational voters. For instance, the "mix-and-match" approach of Table 5 shows that a procedure satisfying the triplet IIA condition and anonymity cannot distinguish the transitive Condorcet profile $a \succ b \succ c \succ d$, $b \succ c \succ d \succ a$, $c \succ d \succ a \succ b$, $d \succ a \succ b \succ c$ from the nontransitive four-voter profile

1, 2	$a \succ b \succ c$	$b \succ c \succ d$	$c \succ d \succ a$	$d \succ a \succ b$
3	$b \succ c \succ a$	$c \succ d \succ b$	$d \succ a \succ c$	$a \succ b \succ d$
4	$c \succ a \succ b$	$d \succ b \succ c$	$a \succ c \succ d$	$b \succ d \succ a$

where voters one and two have the cyclic preferences and voters two, three and four define a Condorcet cycle over each triplet (so, with a positional method, their preferences cancel). This inability to distinguish rational from confused voters suggests that positional election outcomes over the triplets need not correspond to the same profiles positional election outcome for all four candidates. (While my proof uses radically different arguments, this is the case; see (Saari 1995d) and the cited references.) A cause of these paradoxes, then, is the loss of transitivity.

Because the spirit of IIA is so prevalent in decision theory and economics, we must expect many consequences of this unintended violation of transitive preferences. For instance, when a profile defines the conflicting $a \succ b \succ c \succ d$ and $c \succ b \succ a$ election rankings, we must wonder whether a (not c because it comes from a group ranking that may reflect a loss of transitivity) is the voters' preferred candidate. This assertion questions the rationale for runoff elections.

More generally, whenever a procedure ignores some of the agents' available substitutes without relating them, transitivity must be assumed to be lost – the actual (rather than the intended) domain for the specified procedure includes an unexpected variety of perverse preferences. Notice how this situation extends outside of choice theory to include topics from, say, probability where probabilities replace preferences (Saari, 1991, 1995d), statistics, some game theory issues (Saari, 1991), topics in economics such as consumer surplus, or, more generally, the behavior of the aggregate excess demand functions for the different subsets of commodities (where each agent holds fixed his or her holding of the goods not in that subset). (This argument introduces a significantly different interpretation for the last two sections of (Saari, 1995c) and the cited references.) So, the new critical issue is to determine whether imposing the profile restriction of transitive preferences suffices for a procedure to distinguish between transitive and nontransitive profiles. Because many procedures in actual use cannot make these distinctions, difficulties must be anticipated. (New implications for positional voting obtained by examining consequences of this statement will be reported elsewhere.)

To reclaim the lost transitivity, the rankings of subsets must include other available substitutes; that is, resolutions of these problems require appropriate connectivity conditions. To explain, because IIA requires the ranking of each pair to be done totally independent of the ranking of any other pair, the procedures are denied any ability to sequence the pairwise outcomes. Thus, the effect of IIA resembles having separate committees, in total isolation of each other, making partial conclusions for the full group; only chance will permit rational final outcomes. But, with communication and coordination among the committees, reason can be expected. Similarly, if IIA is replaced by a condition allowing the group's relative ranking of $\{a, b\}$ to involve not only each voter's $\{a, b\}$ ranking, but also how each voter ranks all pairs involving a or b, then, as indicated below, the BC is an admissible procedure. By violating IIA, the connectivity conditions allow rational outcomes.

To end this section with some technical comments, observe that the Pareto condition is a mere convenience for Sen's or Arrow's conclusion; it can be replaced with any condition ensuring that the outcomes for enough pairs can change. For instance, for Sen's assertion with $k = 3$ the decisive voters already supply the necessary change. This is because for any $\{b, c\}$ outcome, binary preferences can be chosen for the decisive voters to create a cycle; Prop. 1 ensures that this partial profile is supported by indistinguishable transitive and cyclic profiles. The extension of Arrow's Theorem in (Saari, 1994, 1995b) replaces (P) with an "involvement" condition that (for $k = 3$) only requires the rankings of at least two pairs to change.

Similarly, the only need (in the proof) for transitive outcomes is that any $\{a, b\}$ binary ranking coupled with a particular $\{b, c\}$ ranking uniquely dictates the transitive $\{a, c\}$ ranking; moreover, this condition holds when the pairs are interchanged in any manner. (See Footnote 6.) Arrow's assertion extends to any setting with this functional relationship, so his conclusion holds when indifference is allowed, for set valued mappings, for certain game theoretic analysis, etc. (See Saari, 1991.) Relaxing transitivity to include, say, quasitransitive outcomes provides flexibility because the $a \sim b$ ranking combined with any $\{b, c\}$ ranking fails to completely specify the $\{a, c\}$ ranking; this added freedom leads to Gibbard's (1969) oligarchy conclusion. The reason Gibbard's conclusion remains unacceptable for modern society is that IIA only admits procedures intended for highly unsophisticated primitive voters who can only rank each pair of alternatives. Again, we encounter the notion that if we must build a vehicle using only oxen and carts, do not expect a Porche. Similarly, if we can only use crude unsophisticated procedures, do not expect rational outcomes.

5. Resolutions

By understanding the source of Arrow's and Sen's theorems, it becomes easy to find resolutions. But rather than offering any "best resolution," my goal is to show how this mathematical analysis helps *find* resolutions. Namely, rather than worrying whether conditions are "interpersonal" or satisfy some other condition, the more pragmatic, immediate concern is to identify approaches which can free us from the frustrating world of dictators and impossibility assertions.

To start, it now should be clear that using profile restrictions is not an effective approach. This is because an effective restriction must eliminate all IIA and ML equivalence classes that have a nontransitive profile where "fairness" (e.g., the assumption of (P)) demands a nontransitive outcome. Indeed, the approach used in this paper shows that this is an appropriate way to interpret profile restrictions; e.g., restrictions such as Black's single-peakedness or the replicated preferences common in economic models should be viewed as conditions where the decomposed profiles cannot be reconstructed into profiles with a dominate number of nontransitive preferences (Saari, 1995b). Unfortunately, as true with ML and $k = 3$, effective profile restrictions can be sufficiently severe to kill any interest in the resulting system.

A more serious objection to profile restrictions is that we still are trying to address the needs of a sophisticated society with highly unsophisticated procedures. Thus we must anticipate highly stilted methods with no practical interest. The truth of this assertion becomes apparent by examining the warped procedures required by profiles restrictions; e.g., see (Kalai, Muller, 1977), (Kalai, Ritz, 1980), (Gibbard, Hylland, Weymark, 1987), and, for the weakest possible profile restriction (where only one voter is restricted from just one ranking) (Saari, 1991, 1995b).

An important lesson learned from the approach introduced here is that although IIA and related concepts such as ML seem attractive, their hidden cost of lost transitivity renders them unrealistic and unusable. We now know that if transitive outcomes are desired, subsets of alternatives cannot be separated from available substitutes; IIA needs to be replaced with conditions which introduce connectivity and comparisons.

This makes sense; even a first course on choice theory emphasizes the sequencing conditions that pairwise rankings must satisfy to model rational behavior. The same analysis should be applied to the design of procedures; we need conditions which appropriately sequence the pairwise outcomes. Simply stated, if we want rational outcomes, we must use methods specifically intended for rational agents!

I indicate how to do this by showing that a slight change in Arrow's IIA assumption leads to a positive conclusion. We now know that something must be added to IIA so that the modified axiom recognizes transitive preferences. So, what kind of added information will ensure that the voter is rational? When posed in this manner, answers are immediate.

To illustrate, if a $a \succ b$ ranking comes from a transitive preference, then we can specify how many candidates separate a and b in the transitive ranking – call this the *intensity level* (Saari, 1994, 1995a, b). For instance, the intensity level is zero if a and b are ranked next to each other as in $a \succ b \succ c \succ d$ or in $c \succ a \succ b \succ d$, but the intensity level is two for the $a \succ c \succ d \succ b$ ranking. Notice that the intensity level is zero for each pair ranked by an irrational voter capable only of ranking pairs. As this "intensity of preferences" is a weak way to identify when we are dealing with a society of rational voters, it is worth determining whether it allows reasonable procedures.

To illustrate how even a minimal amount of rationality admits ND procedures, for $k = 3$ replace IIA with

(IIA*) (IIA applies to all voters except voter-one. For voter-one let IIA apply except for $\{a, b\}$ rankings where the intensity level is specified.)

According to Sect. 2 we must determine how IIA* changes the actual domain and the class of admissible procedures. It is easy to see that the IIA* effective domain allows voters 2 to n to have no sequencing abilities with their binary rankings, but voter-one must be at least partially rational. The reward is that, in addition to the IIA procedures designed only for $B^n(3)$, there are some new nondictatorial procedures which exploit this partial rationality. One class, for instance, requires voter-one to be a dictator over $\{a, c\}$, $\{b, c\}$ rankings, and a dictator over $\{a, b\}$ rankings when this ranking is level-one. When voter-one's $\{a, b\}$ ranking is level-zero, the other voters can rank $\{a, b\}$ with, say, a majority vote.

The reason this nondictatorial procedure is undesirable is that IIA* still admits only methods designed for unsophisticated voters. Here, the partially rational voter is a partial dictator and the primitive voters can participate only in certain settings and only on one pair. But this is as it should be because IIA* tacitly assumes that these voters are incapable of doing anything more. This suggests that by further relaxing IIA to recognize the rationality of more voters, then more procedures with rational outcomes will be admitted. In fact, with

(IIIA) (*Intensity IIA.* Society's relative ranking of a pair depends only on the voters' relative rankings of this pair *and their intensity levels of this ranking.*)

we finally obtain realistic procedures.

Theorem. *For $k \geq 3$ alternatives and $n \geq 2$ voters with transitive preferences, there exist procedures with transitive rankings that satisfy anonymity, (P), and IIIA.*

One such procedure is the BC (Saari, 1995a); somewhat surprising, the BC is the only positional method admitted by these conditions! A rich selection of other admissible procedures are based on the BC; e.g., a BC runoff, Black's method (where a Condorcet winner is selected when one exists, otherwise the BC winner is chosen), and so forth. If anonymity is relaxed to (ND), then we can admit methods such as a weighted BC system where some voter's vote counts as though they were cast by j voters. The critical point is that by using a condition that restores the transitivity of individual preferences, realistic possibility assertions emerge! This adds important support for the assertion that if you want rational outcomes, you must use procedures designed for rational people.

The only part of the proof of the theorem that needs explanation is to show why the BC satisfies IIIA. To do so, use the alternative BC formulation (Borda 1781, Saari 1995b) where each voter votes on each pair of alternatives; the sum of points a voter gives a candidate over all pairwise election agrees with the BC vale he assigns her. So, a transitive voter with the relative ranking $a \succ b$ assigns one more point to a than b based on the $\{a, b\}$ election. In all other pairwise $\{a, c\}$ and $\{b, c\}$ elections, a receives a point more than b if and only if this voter ranks c between a and b. Thus, IIIA is satisfied, but IIA is not because the $\{a, b\}$ outcome requires information about the voters' $\{a, c\}$ and $\{b, c\}$ beliefs.

While it is easy to find supporting philosophical arguments for IIIA, my sole purpose is to show how to obtain rational outcomes by requiring procedures to recognize transitivity. Indeed, IIIA can be replaced with alternative conditions connecting subsets of alternatives; e.g., we could allow the group ranking for each pair $\{a, b\}$ to depend only on how the voters rank this pair and any other pair containing either a or b. Again, the BC is an admissible procedure, and (using the results of (Saari, 1995b, d)) it is the only admissible positional method.

Thus, one way to resolve the concerns raised by Sen's and Arrow's assertions is to impose conditions that restore the rationality of the voters. This approach also provides a new way to reinterpret earlier attempts to resolve these difficulties; many of them address Sen's Theorem. To illustrate, notice from Sen's proof that the Table 2 transitive preferences $a \succ b \succ c$ prove that voter-one has strong views about voter-two's $\{a, c\}$ choice. Similarly, voter-two's preferences $b \succ c \succ a$, exhibit strong views about voter-one's $\{a, b\}$ choice. Probably motivated by these observations, Blau (1975), Kelly (1976), Sen (1976) and others have examined the "intensity of preference." While their supporting philosophical arguments are interesting, we now know why they work; they force the procedure to recognize the rationality of voters. It is this rationality condition that leads to possibility assertions; the philosophical arguments are embellishments or justifications for accepting these conditions.

So, with Sen's Theorem, we need conditions to restore the assumption that voters are rational – U does not suffice. One approach is to modify ML (e.g., with intensity of preference arguments, modified liberalism (Blau, 1975, Sen, 1976) or ideas of (Gaertner, Pattanaik, and Suzumura, 1992)). Another is to modify (P) (again, see Sen (1976)). If I have the right to decide between a and b, then why can society compare either of these alternatives with others? This suggests relaxing (P) in the following manner.

Modified Pareto (P*). If individual j is decisive over a pair $\{a, b\}$, then (P) holds except for those pairs involving either a or b.

This condition leads to the following possibility theorem.

Theorem. *Assume there are $n \geq 2$ agents with transitive preferences where at least two agents are decisive over mutually distinct sets of alternatives. There exist procedure with acyclic outcomes which satisfy (P*).*

Rather than promoting this statement as a resolution, my intent is to show how the mathematical analysis leads to resolutions. The purpose of this example is to show that by relaxing (P) – a condition connecting pairs – to (P*), procedures are admitted which partially reclaim the lost transitivity. (Other approaches are possible; e.g., by replacing P with unanimity over the full profile, procedures now can examine relationships among pairs; this change allows possibility assertions.)

The proof of the theorem is immediate. Let $\mathcal{A}$ be the set of all alternatives and let $\mathcal{D}$ be the set of alternatives over which some agent is decisive. Use the BC to rank the alternatives $\mathcal{A}\backslash\mathcal{D}$, and let the decisive voters rank the various $\mathcal{D}$ subsets. The rankings among the various subsets of $\mathcal{D}$ and with $\mathcal{A}\backslash\mathcal{D}$ cannot be determined by P*, so the procedure can impose this ranking. To illustrate with the example of Table 2 (which does not satisfy the conditions of the theorem because two decisive agents have a as an alternative in their assigned set), after the two sets are ranked, select the $\{b, c\}$ ranking so it is consistent with transitivity. In Table 3, just assume that the $\{a, b\}$ ranking always is ranked above the $\{c, d\}$ ranking.

To conclude, now that we understand why the impossibility assertions occur – the admissible procedures cannot recognize when voters are rational – it becomes easy to find resolutions. This assertion extends beyond choice theory.

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