Markov Matrix Multiplication

Brian Abram - Math 210 Fall 2001

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Indeed, it is the case that the product of any two Markov matrices yields a Markov matrix. Let us first start with a simple case of a 2×2 matrix.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Now we just have to show that the columns of this product equal one, knowing that $(a+c) = (b+d) = (e+g) = (f+h) = 1$.

$$\begin{aligned} ae+bg+ce+dg &= ae+ce+bg+dg \\ &= e(a+c)+g(b+d) \\ &= (e+g) \\ &= 1 \end{aligned}$$

$$\begin{aligned} af+bh+cf+dh &= af+cf+bh+dh \\ &= f(a+c)+h(b+d) \\ &= (f+h) \\ &= 1 \end{aligned}$$

Let us now formalize this for any Markov matrices. Let A and B be $n \times n$ Markov matrices, and let AB be the product of A and B.

$$\begin{aligned} AB &= \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ \dots & \dots & \dots \\ a_{n,1} & \dots & a_{n,n} \end{bmatrix} \begin{bmatrix} b_{1,1} & \dots & b_{1,n} \\ \dots & \dots & \dots \\ b_{n,1} & \dots & b_{n,n} \end{bmatrix} \\ &= \begin{bmatrix} a_{1,1}b_{1,1}+\ldots+a_{1,n}b_{n,1} & \dots & a_{1,1}b_{1,n}+\ldots+a_{1,n}b_{n,n} \\ \dots & \dots & \dots \\ a_{n,1}b_{1,1}+\ldots+a_{n,n}b_{n,1} & \dots & a_{n,1}b_{1,n}+\ldots+a_{n,n}b_{n,n} \end{bmatrix} \end{aligned}$$

Adding up the columns in the matrix, we can see that these all add up to one. In any column i, we know from before that $(a_{1,i}+\ldots+a_{n,i}) = (b_{1,i}+\ldots+b_{n,i}) = 1$. Therefore:

$$\begin{aligned} a_{i,1}b_{1,i}+\ldots+a_{i,n}b_{n,i}+\ldots+a_{n,1}b_{1,i}+\ldots+a_{n,n}b_{n,i} \\ a_{i,1}b_{1,i}+\ldots+a_{n,1}b_{1,i}+\ldots+a_{i,n}b_{n,i}+\ldots+a_{n,n}b_{n,i} \\ (a_{1,i}+\ldots+a_{n,i})b_{1,i}+\ldots+(a_{1,i}+\ldots+a_{n,i})b_{n,i} \\ b_{1,i}+\ldots+b_{n,i} = 1 \end{aligned}$$