Properties of Tournaments Among Well-Matched Players

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Dedicated to the memories of John Maybee and Norman Pullman.

1. TOURNAMENTS. In an n-player round robin tournament, each player plays one match against each of the other $n - 1$ players. The win-loss outcomes of these matches can be conveniently recorded in a *tournament matrix* $A = [a_{ij}]$ as follows: First label the players in any order as $1, 2, \ldots, n$. For each pair i and j, set $a_{ij} = 1$ if player i defeats player j, and set $a_{ij} = 0$ otherwise. If $i \neq j$, then exactly one of a_{ij} and a_{ji} is nonzero; when $i = j$, $a_{ii} = 0$.

What properties of the matrix A are related to the strengths of the players? The simplest measure of strength is the number of matches that the player wins, and the row sums of A count the number of matches won by each player. We are interested in understanding tournaments among players who are well matched in the sense that each player wins about half of the matches played.

If the number of players is odd, many properties of A are very well understood. If the number of players is even, however, the properties of A are far less well understood. Indeed, there are many easily stated questions that lead to hard, open problems. Some of these problems are the focus of this paper.

We let I denote the identity matrix, we let J denote the square matrix all of whose entries are ones, and we let e denote the column vector whose entries are all ones. A matrix A whose entries are zeros and ones is a tournament matrix exactly when

$$A + A^T = J - I. \tag{1}$$

Thus

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{2a}$$

is the matrix for a tournament in which the first player defeats the second and third players, the second player defeats the fourth player, and so on.

Tournament matrices have been studied extensively; see [**1**]–[**3**], [**5**]–[**7**], [**8**]–[**10**], and [**12**]–[**27**]. Many properties of tournament matrices are immediate from (1). For example, if A is a tournament matrix, then so are A^T, every principal submatrix of A, and PAP^T for all permutation matrices P.

How many different n-player tournaments are there? For each pair of players, there are two choices for the match winner, and since there are $\binom{n}{2}$ distinct pairs of players, it follows that there are $2^{\binom{n}{2}}$ different $n \times n$ tournament matrices. A particular round robin tournament can, however, be represented by many different tournament matrices. Since the construction of a tournament matrix requires ordering the players, listing the players in a different order can produce a different (but permutation similar) matrix. Thus

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \tag{2b}$$

and (2a) are tournament matrices for the same round robin tournament; reorder the players in (2a), listing them in the order 2, 4, 1, 3.

How many truly different n-player round robin tournaments are there and what do they look like? That is, how many permutation similarity classes of tournament matrices are there, and is there a way to find a representative from each class? There is a closed form formula for the number of classes ([**8**], [**22**]); but for $n > 8$, there is neither a list of canonical representatives nor a known efficient algorithm for building such a list. For $n \leq 8$, Table 1 summarizes results from [**25**]; regular and almost regular tournament matrices are defined in Section 4. David Gregory has produced MATLAB m-files that generate a canonical representative for each isomorphism class of $n \times n$ tournament matrices for $n \leq 8$, and for each isomorphism class of 9×9 regular tournament matrices. Gregory's m-files as well as a variety of other m-files for building tournament matrices and computing their properties can be found on-line [**28**].

TABLE 1 The number of classes of nonisomorphic tournament matrices and the number of classes that are either regular (R) or almost regular (AR).

n	$\frac{n(n-1)}{2}$	$2^{\frac{n(n-1)}{2}}$	Classes	R/AR Classes
2	1	2	1	1
3	3	8	2	1
4	6	64	4	1
5	10	$2^{10} = 1024$	12	1
6	15	$2^{15} \doteq 32K$	56	5
7	21	$2^{21} \doteq 2.1M$	456	3
8	28	$2^{28} \doteq 268M$	6880	85

2. RANKING THE PLAYERS.

How do the players compare? The most natural answer is to compare the number of games won by each player. Let A be a tournament matrix. Since $a_{ij} = 1$ exactly when player i beats player j, it follows that the number of games won by player i is just the sum of the entries in the i^{th} row of A. Since $s = Ae$ is the vector of row sums, it is called the *score vector* for A. We say that *player i is stronger than player j* if $s_i > s_j$. Score-vector-based ranking has been investigated in [**5**] and [**19**].

Let U_n denote the unique strictly upper triangular $n \times n$ tournament matrix. That is, U_n is the matrix for a tournament in which the i^{th} player beats the j^{th} player whenever $i < j$. The score vector for U_n is $(n-1, n-2, \ldots, 2, 1, 0)^T$. Every player has a distinct score; and thus the score vector unambiguously ranks the players. The only tournament matrices for which every player has a distinct score are those that are permutation similar to U_n.

What can be said about ranking players in a tournament when two or more players have the same score? This is a more difficult question to answer. Before pursuing this question, we need further terminology.

A square matrix M is called *irreducible* if either M is a 1×1 matrix or else no permutation matrix P exists such that PMP^T is block upper triangular with two or more square, diagonal blocks. Since spectral properties of matrices are preserved under permutation similarity, and since the diagonal blocks of a block triangular matrix determine its spectrum, it is natural to focus our attention on irreducible tournament matrices.

Suppose that A is an $n \times n$ tournament matrix, and that there are a positive integer k and a permutation matrix P such that

$$PAP^T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix},$$

where T_{11} is $k \times k$, and T_{22} is $(n-k) \times (n-k)$. Since A is a tournament matrix, so are PAP^T, T_{11}, and T_{22}; and T_{12} is the $k \times (n-k)$ all ones matrix. The first k scores of PAP^T must be at least $n-k$, and the last $(n-k)$ scores must be at most $n-k-1$. Thus the first k players all have higher scores than the last $n-k$ players, and each of the first k players beats *all* of the last $n-k$ players.

Consequently, if we want to understand how to rank players reasonably in this tournament, it suffices to focus on ranking the players within the two groups corresponding to the two diagonal blocks of PAP^T. If either of these blocks is itself not irreducible, then we can repeat the process of finding a permutation similarity that transforms that diagonal block into a block triangular matrix. Thus, understanding how to rank players in a general tournament reduces to understanding how to rank players in a tournament with an irreducible tournament matrix.

The *spectral radius* of a real, square matrix M, denoted $\rho = \rho(M)$, is the maximum of the absolute values of the eigenvalues of M. A square matrix M is called a *primitive matrix* if all of the entries of M are nonnegative and some positive integer power of M has all entries positive. One of the most famous results in combinatorial matrix theory is the Perron-Frobenius Theorem; see [**11**, Theorems 8.4.4 and 8.5.2].

Theorem 1. (*Perron-Frobenius*). *Let M be an irreducible matrix with all entries nonnegative. Then $\rho(M)$ is a simple eigenvalue for M, and there is an entrywise positive eigenvector v for ρ. Further, any nonnegative eigenvector for M is a multiple of v. Finally, if M is also primitive, every eigenvalue λ of M other than ρ satisfies $|\lambda| < \rho$.*

We call the entrywise positive eigenvector v with euclidean norm $\|v\|_2 = 1$, whose existence and uniqueness are guaranteed by the Perron-Frobenius Theorem, *the Perron vector for M*. Next, we examine the role that the Perron vector plays in ranking the players.

For a tournament matrix A, the product $a_{ij}a_{jk}$ is nonzero exactly when player i defeats player j and player j defeats player k. This suggests that player i is stronger than player j, and that player i is even stronger compared to player k. Similarly, a nonzero product of the form $a_{ij}a_{jk} \cdots a_{hm}$ suggests that player i is much stronger than player m. Unfortunately, for each player i in an irreducible tournament, there is always some nonzero product of the form $a_{ij}a_{jk} \cdots a_{mi}$, which accordingly suggests that player i is much stronger than player i! Nonetheless, the products of entries from A can still play a role in ranking the players.

Suppose we measure the strength of player i by computing the sum of the scores of the players that player i beats: $\sum_{j:\ i \text{ beats } j} s_j$. After all, a strong player should beat players who have, in turn, beaten lots of other players. Since i beats j

exactly when $a_{ij} = 1$, our new strength measure is just $\sum_{j=1}^{n} a_{ij} s_j = \sum_{j=1}^{n} [a_{ij} \sum_{k=1}^{n} a_{jk}] = \sum_{k=1}^{n} \sum_{j=1}^{n} a_{ij} a_{jk}$, which is just the sum of all of the entries in the i^{th} row of A^2. That is, $A^2 e$ is the vector whose i^{th} entry is the sum of the scores of all of the players defeated by player i.

If we think of the score vector Ae, whose entries are the respective numbers of defeated players, as providing a first measure of relative strength, and if we think of the vector $A^2 e$, whose entries are the respective sums of scores of defeated players, as providing a second measure of relative strength, then it is natural to think of the vector $A^3 e$, whose entries are the respective sums of the sums of the scores of defeated players, as providing a third measure of strength. Similarly, we can think of the vectors $A^4 e$, $A^5 e$, or even $A^k e$ for some arbitrary positive integer k, as providing additional measures of relative strength. As k increases, $A^k e$ becomes more sensitive to the relationships among sequences of players, and thus should give a better guide to ranking the players. Since we are interested in measuring the relative strength of the players, and since the entries of $A^k e$ grow quite large as k increases, we can scale the vector $A^k e$ to control the magnitudes of its entries.

Let A be an irreducible tournament matrix. For each positive integer k, let $l_k = \|A^k e\|_2$. Then the sequence of nonnegative vectors $l_1^{-1} Ae$, $l_2^{-1} A^2 e$, $l_3^{-1} A^3 e, \dots, l_k^{-1} A^k e, \dots$ converges to the Perron vector v for A. Why? This is just the *power method*, a well-known iterative technique for finding an eigenvector for the eigenvalue of largest magnitude for a matrix M provided that that eigenvalue is unique [**11**, pp. 62, 523]. Thus the relative sizes of the entries of the Perron vector v for a tournament matrix provide another means for ranking the players in a tournament. This is essentially the approach taken by Kendall [**14**] and Wei [**27**], and this relative strength ranking is sometimes called the *Kendall-Wei ranking*. Ideally, we would hope that for an irreducible tournament matrix A, if $s_i > s_j$ then $v_i > v_j$. However, this is not generally true! An infinite family of counterexamples can be found in [**5**]. In Section 4, we discuss a class of tournament matrices for which the ranking induced by v is consistent with the ranking based on scores.

If we want to examine tournaments between players who are well matched using the Kendall-Wei ranking, then we want to focus on irreducible tournament matrices for which the entries of v are close to one another. One way to measure how close together the entries are is to compute

$$\begin{aligned} var(v) &\equiv \sum_{i<j} (v_i - v_j)^2 = \sum_{i<j} \left(v_i^2 + v_j^2\right) - 2\sum_{i<j} v_i v_j \\ &= (n-1)\sum_{h=1}^{n} v_h^2 - 2\sum_{i<j} v_i v_j = (n-1) v^T v - \sum_{i \neq j} v_i v_j \\ &= (n-1) v^T v - v^T (J - I) v. \end{aligned}$$

Applying (1) yields

$$\begin{aligned} v^T (J - I) v &= v^T (A + A^T) v = v^T (Av) + (Av)^T v \\ &= v^T (\rho v) + (\rho v)^T v = 2\rho v^T v. \end{aligned}$$

Finally, $v^T v = 1$, so $var(v) = n - 1 - 2\rho$.

Since $var(v) \geq 0$, for an irreducible tournament matrix it follows that $\rho \leq (n-1)/2$; that as ρ increases to $(n-1)/2$, $var(v)$ decreases to zero; and that $\rho = (n-1)/2$ exactly when v is a multiple of e. Thus the tournaments whose players are most well-matched based on the Kendall-Wei ranking are those for which $Ae = (n-1)e/2$. Since the score vector Ae has integer entries, this can happen only when n is odd. What happens when n is even is the subject of Section 4.

A parallel development can be made for the column sums of A and the left Perron vector w of A, which satisfies $w^T A = \rho w^T$ with w entrywise positive. This leads to an analogous relationship between increasing ρ and decreasing $var(w)$, with ρ maximized when n is odd and w is a multiple of e. Since column sums count the number of losses, the entries of w are a measure of the relative weaknesses of the players.

Various other ranking schemes have been proposed. For example, Ramanujacharyula **[24]** ranked players using the ratios of relative strength to relative weakness. For a bibliography of other schemes, consult **[22]**.

Motivated in part by the role that ρ plays, we next discuss the spectral properties of tournament matrices.

3. SPECTRAL PROPERTIES OF TOURNAMENT MATRICES. The following theorem summarizes the eigenproperties of tournament matrices, and is based on results from **[1]**, **[7]**, **[20]**, and **[21]**, and on the Perron-Frobenius Theorem:

Theorem 2. *Let A be an $n \times n$ tournament matrix. Let α and β be eigenvalues for A. Then*:

1. $-1/2 \leq \mathrm{Re}(\alpha) \leq (n-1)/2$;
2. *If* $\mathrm{Re}(\alpha) = -1/2$, *then the algebraic multiplicity and geometric multiplicity of α are equal*;
3. *If* $\mathrm{Re}(\alpha) > -1/2$, *then the geometric multiplicity of α is one*;
4. *If* $\mathrm{Re}(\alpha) = -1/2$, *and if $\beta \neq \alpha$, then every eigenvector for α is orthogonal to every (generalized) eigenvector for β.*
5. *The cyclic subspace spanned by $\{e, Ae, A^2e, \ldots, A^{n-1}e\}$ is the span of the set of (generalized) eigenvectors for A corresponding to eigenvalues with real part greater than* $-1/2$.
6. *A has a nonnegative eigenvalue that is greater than or equal to the modulus of every other eigenvalue, and A has a nonnegative eigenvector for that eigenvalue.*

The cyclic subspace spanned by $\{e, Ae, A^2e, \ldots, A^{n-1}e\}$ is called the *walk space of A* and is denoted by W_A. It follows from properties (2) and (4) that $(W_A)^{\perp}$ is the span of the eigenvectors with real part equal to $-1/2$. It follows from property (3) that if a tournament matrix has a repeated eigenvalue with real part greater than $-1/2$, then the tournament matrix cannot be diagonalizable. It follows from properties (2) and (3) that isospectral tournament matrices (i.e., those with the same characteristic polynomial) must be similar since they must have the same Jordan block structure. Property (6) can be deduced from Theorem 1.

Among all $n \times n$ tournament matrices, the tournament matrices with the smallest spectral radius and the longest Jordan chain are those that are permutation similar to the strictly upper triangular tournament matrix U_n. For U_n, the spectral radius is zero, and the Jordan chain for zero has length n.

In **[16]**, Kirkland proves the following result about irreducible tournament matrices with smallest spectral radius:

Theorem 3. *Among all $n \times n$ irreducible tournament matrices with $n \geq 3$, the tournament matrices with the smallest spectral radius are those that are permutation similar to the tournament matrix obtained from U_n by setting $u_{j,j+1} = 0$ and $u_{j+1,j} = 1$ for all j with $1 \leq j \leq n-1$.*

The score vector for the $n \times n$ irreducible, spectral radius minimizer is $[n-2, n-2, n-3, \dots, 2, 1, 1]^T$, and the Perron vector v, which gives the Kendall-Wei relative strengths, satisfies $v_2 > v_1 > v_3 > v_4 > \cdots > v_{n-1} > v_n$.

We return to the question of spectral radius maximizers in subsequent sections.

4. REGULAR AND ALMOST REGULAR TOURNAMENT MATRICES. We now return to tournaments between well-matched players. We are interested in tournaments in which all of the players have scores as close to equal as possible. When n is odd, there exist tournaments for which all players have the same score: $(n-1)/2$. These tournaments and their corresponding tournament matrices are called *regular*. When n is even, $(n-1)/2$ is not an integer, so the closest we can get to all players having the same score is for half of the players to have score $\lfloor (n-1)/2 \rfloor = (n-2)/2$ and half to have score $\lceil (n-1)/2 \rceil = n/2$. Such tournaments and their corresponding tournament matrices are called *almost regular*. Since players can be ordered arbitrarily, we can assume that almost regular tournament matrices always have their first $n/2$ row sums equal to $\lfloor (n-1)/2 \rfloor$. The matrices A and B given in Section 1 are almost regular. Kirkland has shown **[19]** that for almost regular tournament matrices, the Kendall-Wei ranking, the relative weakness ranking, and the Ramanujacharyula ratio ranking are all consistent with the score ranking. That is, if $s_i = n/2$ and $s_j = (n-2)/2$, then $v_i > v_j$, $w_i < w_j$, and $v_i/w_i > v_j/w_j$.

The last column of Table 1 gives the number of distinct nonisomorphic regular/almost regular tournament matrices for $n \le 8$. It is known that when $n = 9$, there are exactly 15 distinct nonisomorphic classes of regular tournament matrices **[28]**. There are far fewer nonisomorphic regular tournament matrices for $n = 9$ than there are nonisomorphic almost regular tournament matrices for $n = 8$. This reflects the fact that there is more symmetry in regular tournament matrices than in almost regular tournament matrices. A natural, open question is: For a general n, how many isomorphism classes of regular or almost regular tournament matrices are there, and how does one produce efficiently a list of class representatives?

We finish this section by noting that regular and almost regular tournament matrices have some nice properties.

Theorem 4. *Regular and almost regular tournament matrices are both irreducible and nonsingular for $n > 2$. Regular and almost regular tournament matrices are primitive for $n > 3$. The spectral radius of an $n \times n$ regular tournament matrix is $(n-1)/2$ and every other eigenvalue has real part $-1/2$. For $n > 2$, the spectral radius of an $n \times n$ almost regular tournament matrix exceeds $(n-2)/2$ and every other eigenvalue has negative real part. When $n = 2$, the unique (up to permutation similarity) tournament matrix is the strictly upper triangular matrix U_2. When $n = 3$, the unique (up to permutation similarity) irreducible tournament matrix is*

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Proof: It is easy to see that when $n = 2$, the unique (up to permutation similarity) tournament matrix is U_2. It is easily checked that the only 3×3 irreducible tournament matrices are the permutation matrices corresponding to 3-cycles, all of which are regular and permutation similar. In **[23]**, Moon and Pullman prove that for $n > 3$, irreducible tournament matrices are primitive.

Suppose that the $n \times n$ tournament matrix A is either regular or almost regular. To prove irreducibility for $n > 2$, assume to the contrary that there exist a permutation matrix P and positive integers p and q with $p + q = n$ such that PAP^T is a block upper triangular, tournament matrix:

$$PAP^T = \begin{bmatrix} M & J_{p \times q} \\ 0_{q \times p} & N \end{bmatrix}.$$

Since A is either regular or almost regular, every row sum of PAP^T is in the closed interval $[(n-2)/2, n/2]$. If $q = 1$, then the last row of PAP^T has row sum zero. Since $n > 2$, this contradicts the lower bound on the row sums. So assume $q \geq 2$. Since M is a $p \times p$ tournament matrix, M must contain a total of $p(p-1)/2$ ones. Then some row of M must contain at least $(p-1)/2$ ones. The corresponding row of PAP^T then has row sum at least $(p-1)/2 + q = (p+q)/2 + (q-1)/2 > n/2$, which contradicts the upper bound on the row sums. Thus A must be irreducible.

Let A be either regular or almost regular with $n > 2$. Let ρ be the spectral radius of A. Since A is irreducible, it follows from Theorem 1 applied to A^T that A has a strictly positive row eigenvector w for ρ. Since w is strictly positive, and since all row sums of A are at least $(n-2)/2$ and at least one row sum exceeds $(n-2)/2$, it follows that $\rho w^T e = w^T(Ae) > w^T((n-2)/2)e$. Thus $\rho > (n-2)/2$.

Let λ_j, $2 \leq j \leq n$, be the remaining eigenvalues of A. By Theorem 2, $\mathrm{Re}(\lambda_j) \geq -1/2$ for each j. Since the trace of A is the sum of the eigenvalues of A, and since every diagonal entry of A is zero, it follows that

$$0 = trace(A) = \rho + \sum_{j=2}^{n} \mathrm{Re}(\lambda_j) \geq \rho + \mathrm{Re}(\lambda_k) + (n-2)(-1/2) \qquad (2)$$

for each k with $2 \leq k \leq n$. Since $\rho > (n-2)/2$, (2) yields $0 > \mathrm{Re}(\lambda_k)$. Thus A must be nonsingular. Further, when A is regular, using $\rho = (n-1)/2$ in (2) together with $\mathrm{Re}(\lambda_k) \geq -1/2$ yields $\mathrm{Re}(\lambda_k) = -1/2$. ∎

5. SPECTRAL PROPERTIES OF REGULAR TOURNAMENT MATRICES. The following theorem summarizes the key spectral properties of regular tournament matrices.

Theorem 5. *Let A be an $n \times n$ tournament matrix. The following are equivalent:*

1. *A is regular.*
2. *$Ae = \lambda e$ for some complex λ;*
3. *$Ae = (n-1)e/2$;*
4. *$AJ = JA$;*
5. *A is normal ($AA^T = A^TA$);*
6. *$J = p(A)$ for some polynomial $p(x)$;*
7. *A is unitarily diagonalizable.*

Proof: 1 ⇔ 6 may be found in [**4**]. 2 ⇔ 3 follows from the fact that A is a matrix of zeros and ones and that e is an all ones vector. Note that $AA^T = A(J - I - A)$ and $A^TA = (J - I - A)A$. Thus $AA^T = A^TA$ if and only if $AJ = JA$, that is, if and only if A is regular [**6**]. 5 ⇔ 7 is a basic fact [**11**, Theorem 2.5.4]. ∎

The next result, which follows from the analysis of $var(v)$ in Section 2, was proved by Brauer and Gentry [1].

Theorem 6. *Among all $n \times n$ tournament matrices with n odd, the tournament matrices with maximum spectral radius are precisely the regular tournament matrices, and all of these tournament matrices have spectral radius $(n-1)/2$.*

6. SPECTRAL PROPERTIES OF ALMOST REGULAR TOURNAMENT MATRICES. Because of the analogous roles played by regular and almost regular tournament matrices, one would expect that the results of Section 5 would have natural analogs. Here, we start to discover that almost regular tournament matrices are much harder to work with than regular tournament matrices. The key differences are that if A is almost regular, then e is not an eigenvector for A, and that J and A do not commute. In contrast to the situation for regular tournament matrices, there are almost regular tournament matrices for which every eigenvalue has real part greater than $-1/2$, e.g., the Brualdi-Li matrices B_{2n} introduced in Section 7. Further, almost regular tournament matrices are never unitarily diagonalizable, and it is not known if they are always diagonalizable. The relationship between almost regularity and the maximization of the spectral radius is quite complicated. In light of these difficulties, we examine a restricted class of almost regular tournament matrices that contains the matrices B_{2n}.

Let A be any $n \times n$ tournament matrix. From A, we construct the $2n \times 2n$ almost regular tournament matrix

$$M_A \equiv \begin{bmatrix} A & A^T \\ A^T + I & A \end{bmatrix}.$$

Since $A + A^T = J_n - I_n$, it is clear that the first n rows of M_A have row sum $n-1$, and that the last n rows of M_A have row sum n.

How are the eigenstructures of A and M_A related? For any tournament matrix A, we know that the eigenvectors for eigenvalues with real part equal to $-1/2$ span $(W_A)^\perp$. The following result from [9] constructs $(W_{M_A})^\perp$ from $(W_A)^\perp$.

Theorem 7. *Let A be an $n \times n$ tournament matrix. Then:*

1. $\dim(W_{M_A}) = 2\dim(W_A)$;
2. $\dim(W_{M_A})^\perp = 2\dim(W_A)^\perp$;
3. *w is an eigenvector of M_A corresponding to an eigenvalue γ with* $\mathrm{Re}(\gamma) = -1/2$ *if and only if*

$$w = \begin{bmatrix} \sqrt{\lambda}\, u \\ \pm\sqrt{-\lambda}\, u \end{bmatrix},$$

where u is an eigenvector of A corresponding to the eigenvalue λ with $\mathrm{Re}(\lambda) = -1/2$. *When this is the case, $\gamma = \lambda \pm \sqrt{-\lambda\bar{\lambda}}$.*

Thus the diagonalizability of M_A is determined in some sense by how the vectors in W_A give rise to vectors in W_{M_A}. In one special case [9], we have

Theorem 8. *Let A be a regular tournament matrix. Then M_A is diagonalizable, and if A has k distinct eigenvalues then M_A has $2k$ distinct eigenvalues.*

It is not known if M_A is diagonalizable for every tournament matrix A.

7. BRUALDI-LI TOURNAMENT MATRICES. In this section we focus on tournament matrices M_A where $A = U_n$; these are the *Brualdi-Li matrices*, and are denoted B_{2n}. The matrix (2b) is evidently B_4. The Brualdi-Li matrices have been the subject of much study: see **[3]**, **[9]**, **[10]**, **[17]**, and **[18]**.

First, we give a simple test that determines if a given almost regular tournament matrix is permutation similar to a Brualdi-Li matrix.

Theorem 9. *Let A be an almost regular, $2n \times 2n$ tournament matrix. Let s denote the score vector of A. Let Λ denote the index set for the rows with row sum $n - 1$. Let Γ denote the index set for the rows with row sum n. The tournament matrix A is permutation similar to B_{2n} exactly when*

$$s_\Lambda^T s_\Lambda = s_\Gamma^T s_\Gamma = 2\binom{n}{3} + \binom{n}{2}.$$

Proof: Note that $\Gamma = \{1, 2, \ldots, 2n\} \setminus \Lambda$. Then $A[\Lambda, \Lambda]$ and $A[\Gamma, \Gamma]$, the principal submatrices of A indexed by the sets Λ and Γ, respectively, must be tournament matrices. Since rows with indices in Λ all have row sum $n - 1$ and since the rows with indices in Γ all have row sum n, any permutation similarity between A and B_{2n} must carry the rows indexed by Λ to rows of B_{2n} with row sum $n - 1$. Now A is itself permutation similar to

$$\begin{bmatrix} A[\Lambda, \Lambda] & A[\Lambda, \Gamma] \\ A[\Gamma, \Lambda] & A[\Gamma, \Gamma] \end{bmatrix}.$$

By the first corollary on page 9 of **[22]**, if B is an $n \times n$ tournament matrix with score vector s_B, then $s_B^T s_B \leq 2\binom{n}{3} + \binom{n}{2}$, with equality exactly when B is permutation similar to U_n. Given the hypotheses on s_Λ and s_Γ, it follows that $A[\Lambda, \Lambda]$ and $A[\Gamma, \Gamma]$ are both permutation similar to U_n. Thus A must be permutation similar to

$$\begin{bmatrix} U_n & C \\ J - C^T & U_n \end{bmatrix}$$

for some (0, 1)-matrix C. Now by **[10**, Lemma 3.18], any almost regular tournament matrix of this form must in fact have $C = U_n^T$, and hence A must be permutation similar to B_{2n}. ■

In previous sections, we asked about the diagonalizability of almost regular tournament matrices in general, and of almost regular tournament matrices of type M_A in particular. Here, we focus on a single almost regular matrix, and ask the open question: Is B_{2n} diagonalizable? Computational evidence suggests that the answer is yes; this has been checked using MATLAB for $n \leq 25$. Since the walk space of U_n is n-dimensional, the walk space of B_{2n} must be $2n$-dimensional by Theorem 7.

We close with two results and a conjecture that give further reasons why the Brualdi-Li matrices are worthy of study. These are closely related to Theorem 6, which states that regular tournament matrices are spectral radius maximizers. The first theorem is restricted to tournament matrices of type M_A. The second theorem is an analog to Theorem 6, and motivates our interest in almost regular tournament matrices. These theorems indicate that almost regular tournament matrices are central to determining which tournaments have players that are the most evenly matched in the sense of minimizing the variation in the Kendall-Wei ranking vector. The conjecture, known as the Brualdi-Li Conjecture [3], claims that among all tournaments with an even number of players, the tournament corresponding to the Brualdi-Li matrix has the most well-matched players in the sense of minimizing the variation in their Kendall-Wei ranks.

Theorem 10. [9] *Among all $2n \times 2n$ tournament matrices of the type M_A, the tournament matrices with maximum spectral radius are all permutation similar to the Brualdi-Li matrix B_{2n}.*

Theorem 11. [18] *Among all $2n \times 2n$ tournament matrices for n sufficiently large, the tournament matrices with maximum spectral radius are almost regular.*

Conjecture 12. [13] *Among all $2n \times 2n$ tournament matrices, the tournament matrices that maximize the spectral radius are permutation similar to B_{2n}.*

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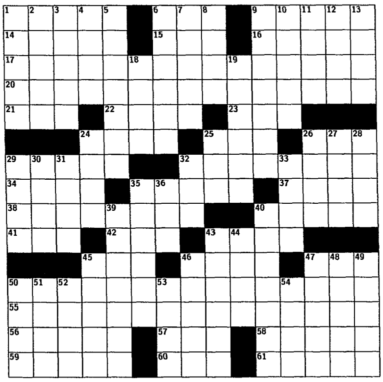
Advanced Calculus Theorem



ACROSS

1 French dieresis
6 "The racer's edge"
9 Invitee of 44 down
14 Macho action figure
15 Raven author
16 ______ World Turns (soap opera)
17 Start of the theorem
20 Second part of the theorem
21 Edward for short
22 Cay
23 TAMU airport symb.
24 Abbreviated quantities
25 ______ loves you (Beatles song)
26 In the groove
29 Halt
32 Sugar-free drink
34 Prefix with derivative
35 Square and figure-eight
37 Xmas
38 Props for Tiger Woods
40 Sir's counterpart
41 Female sheep
42 Palindromic Nigerian state
43 Ocean vessel
45 Opposite of 34 across
46 City of David
47 AIDS virus
50 Third part of the theorem
55 End of the theorem
56 Colored steel by heating
57 Mimic
58 Adolescents
59 Subway
60 Napoleon's bravest general
61 Shopping binge

DOWN

1 Larceny
2 Musical show
3 Correct
4 Cleopatra's Antony
5 ______ at all (Beatles song)
6 Cylinders of thread
7 1,000 kilograms
8 Radical animal rights grp.
9 Armored glove
10 Customary
11 FGH path?
12 Avoid
13 Thomas Hardy's Pure Woman
18 $\exp(it)$ variant
19 Chiggers cause them
24 "No way!"
25 Perch
26 Gangster
27 Concept
28 Conceal
29 Pen
30 Poetic enough
31 Fields medalist Selberg
32 Windows predecessor
33 ______ , crackle, and pop
35 Signature with no sharps or flats
36 New combining form
39 Twister
40 Stately dances
43 Actor Poitier
44 Entertainer of 9 across
45 Bag musician
46 Shoot from hiding
47 Float
48 Pointless
49 French waltz
50 Guided missile
51 Decorative painting
52 Indecent material
53 MATLAB's inverse tangent
54 Type of school for 58 across

Contributed by Harold Boas, Texas A & M University