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\documentclass[11pt]{article}

\usepackage{amsmath}
\usepackage{amssymb}

%           *** CHANGE DIMENSIONS ***
\voffset=-0.3truein     % LaTeX has too much space at page top
\addtolength{\textheight}{0.3truein}
\addtolength{\textheight}{\topmargin}
\addtolength{\topmargin}{-\topmargin}
\textwidth  6.0in       % LaTeX article default 360pt=4.98''
\oddsidemargin 0pt  % \oddsidemargin  .35in   % default is 21.0 pt
\evensidemargin 0pt % \evensidemargin .35in   % default is 59.0 pt
\mathsurround 1pt

\newcommand{\R}{\mathbb{R}}
\newcommand{\C}{\mathbb{C}}
\newcommand{\vu}{\vec{u}}
\newcommand{\vU}{\vec{U}}
\newcommand{\vv}{\vec{v}}
\newcommand{\vw}{\vec{w}}
\newcommand{\abs}[1]{\lvert #1 \rvert}       % absolute value
\newcommand{\norm}[1]{\lVert #1 \rVert}      % norm
\newcommand{\ip}[2]{\langle #1,\, #2\rangle} % ip = inner product

%==================== END PREAMBLE =====================
\begin{document}
\parindent=0pt
\vspace*{ -2cm}
{Math 202 Fall 2013 \hfill Jerry Kazdan}

\smallskip
\begin{center}
{\large\bf ODE: Existence and Uniqueness of a Solution}
\end{center}

\medskip
The Fundamental Theorem of Calculus tells us how to solve the ordinary
differential equation (ODE)
\[
\frac{du}{dt} = f(t) \qquad\text{with initial condition }\qquad
u(0)=\alpha.
\]
Just integrate both sides:
\[
u(t)=\alpha +\int_0^t f(s)\,ds.
\]
It is not obvious how to solve
\[
\frac{du(t)}{dt} = f(x,u(t)) \qquad\text{with initial condition }\qquad
u(0)=\alpha
\]
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because the unknown, $u(t)$, is on both sides of the equation.  In many
particular cases, by using special devices one can find formulas for
the solutions -- but it is far from obvious that a solution exists or
is unique.  In fact, there are simple examples showing that unless one
is careful, a solution may not exist, and even if one exists, it may
not be unique.  Just because one may want something to happen doesn't
mean that this will happen.  It is easy to have presumptions that turn
out to be not quite true.

\smallskip
The results that we present are classical.  They will use most of the
ideas we have covered this semester.

\medskip
We investigate one illuminating case and will prove the existence and
uniqueness of a solution of the system of inhomogeneous linear equation
\begin{equation}\label{ode1}
\frac{d\vU(t)}{dt} = A(t)\vU(t) +\vec{F}(t) \qquad\text{with}\qquad
\vU(0)=\vec{\alpha}.
\end{equation}
Here we seek a vector
$\vU(t)=\begin{pmatrix}u_1(t)\\\vdots\\u_n(t))\end{pmatrix}$ given
the input $n \times n$ matrix  $A(t)=(a_{ij}(t))$, and a vector
$\vec{F}(t)=\begin{pmatrix}f_1(t)\\\vdots\\f_n(t))\end{pmatrix}$. The
elements of $A(t)$ and $F(t)$ are assumed to  depend continuously on
$t$ for $\abs{t}\le b$.  Also $\alpha\in\R^n$ is the initial condition.

\medskip
Although we will not pursue it, there is fairly straightforward
extension of the method we use to the more general nonlinear case
\[
\frac{d\vU}{dt} = \vec{F}(t,\,\vU(t))\qquad\text{with}\qquad
\vU(0)=\vec{\alpha}.
\]
The ideas are already captured in our special case of equation
\eqref{ode1}.

\medskip
{\bf Example } Here we have a system of two equations
\[
\begin{aligned}
u_1'(t) =&-u_2(t)\\ u_2'(t)=&\phantom{-}u_1(t)\end{aligned}
\]
with initial conditions $u_1(0) = 1$ and $u_2(0)=0$. The (unique)
solution of this happens to be $u_1(t)=\cos t$, $u_2(t)=\sin t$, but
the point of these notes is to consider equations where there {\it not}
be simple formulas for the solution.

\bigskip
The primary reason we  are presenting the more general matrix case
$n\ge 1$ is apply to the standard second order scalar initial value
problem
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\begin{equation}\label{ode2}
y''(t)+p(t)y'(t)+q(t)y(t)=f(t) \quad\text{with}\quad y(0)=a
\text{\quad and\quad} y'(0)=b,
\end{equation}
where $p(t)$, $q(t)$, and $f(t)$ are continuous real-valued functions.

To reduce the problem \eqref{ode2} to problem \eqref{ode1}, let
$u_1=y$ and $u_2=y'$. then
\begin{equation*}
\begin{aligned}
u_1' =& \ y'=u_2\\
u_2' =& \ y''=-pu_2-qu_1 + f
\end{aligned}
\end{equation*}
that is,
\begin{equation}\label{ode2a}
\begin{pmatrix} u_1\\u_2\end{pmatrix}'
=\begin{pmatrix}\phantom{-}0 & \phantom{-}1\\ -q & -p\end{pmatrix}
\begin{pmatrix}u_1\\u_2\end{pmatrix} + \begin{pmatrix}0\\f\end{pmatrix}
\quad\text{with}\quad\begin{pmatrix} u_1(0)\\u_2(0)\end{pmatrix}
=\begin{pmatrix} a\\b\end{pmatrix}.
\end{equation}
This exactly has the form of the system \eqref{ode1}.  If we can solve
the system \eqref{ode2a}, then $u_1(t)$is the solution of equation
\eqref{ode2}

\bigskip
{\bf Example} Before plunging ahead to prove that equation
\eqref{ode1} has exactly one solution, we present the idea with the
simple example
\begin{equation} \label{exp}
u'=u\qquad\text{with}\qquad u(0)=1
\end{equation}
whose solution we already know is $u(t)=e^t$.  The first step is to
integrate both sides of equation \eqref{exp} to obtain the equivalent
problem of finding a function $u(t)$ that satisfies
\begin{equation}\label{exp2}
u(t)= 1 +\int_0^t u(s)\,ds.
\end{equation}
We will solve this by \emph{successive approximations}.  Let the initial
approximation be $u_0(t)=u(0)=1$ and define the subsequent
approximations by the rule
\begin{equation}\label{exp3}
u_{k+1}(t)=1 + \int_0^t u_k(s)\,ds, \qquad k=0, 1, 2, \dots .
\end{equation}
Then
\[
\begin{aligned}
u_1(t)=& 1+ \int_0^t 1\,ds = 1+t\\
u_2(t)=& 1+ \int_0^t (1+s)\,ds = 1 + t +\tfrac{1}{2}t^2\\
u_3(t)=& 1 + \int_0^t(1+s +\tfrac{1}{2}s^2)\,ds = 1+t+\tfrac{1}{2}t^2 +
\tfrac{1}{3!}t^3\\
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\qquad \vdots & \\
u_k(t)=& 1+t+\tfrac{1}{2}t^2 +\cdots+\tfrac{1}{k!}t^k
\end{aligned}
\]
We clearly recognize the Taylor series for $e^t$ emerging.  This gives
us hope that our successive approximation approach to equation \eqref{exp3}
is a plausible technique.

\bigskip
With this as motivation we integrate equation
\eqref{ode1} and obtain
\begin{equation}\label{int1}
\vU(t)=\vec{\alpha} + \int_0^t\left[A(s)\vU(s) + \vec{F}(s)\right]\,ds
\end{equation}
We immediately observe that if a continuous function $\vU(t)$ satisfies
this, then by the Fundamental Theorem of Calculus applied to the left
side, this $\vU(t)$ is differentiable and is a solution of our equation
\eqref{ode1}.  Therefore we need only find a continuous $\vU(t)$
that satisfies equation \eqref{int1}.

\medskip
As in our example we use successive approximations.  To make the issues
clearer, I will give the proof {\it twice}.  First for the special case $n=1$
\begin{equation}\label{simple0}
u'(t)=a(t)u(t) +f(t) \qquad \text{with}\qquad u(0)=\alpha
\end{equation}
so there are no vectors or matrices.  In this particularly special case
there happens to be an explicit formula for the solution -- but we
won't use it since it will not help for the general matrix case.  We
are assuming that $a(t)$ and $f(t)$ are continuous on the interval
$[0,b]$. Consequently, they are bounded there and we use the uniform
norm
\begin{equation}\label{norm}
\norm{a}:=\sup_{t\in [0,b]}\abs{a(t)}, \qquad
\norm{f}:=\sup_{t\in [0,b]}\abs{f(t)}, \qquad
\end{equation}

Just as with equation \eqref{ode1} we integrate \eqref{simple0} to find
\begin{equation}\label{simple1}
u(t)=\alpha +\int_0^t [a(s)u(s) + f(s)]\,ds.
\end{equation}
As we observed after equation \eqref{int1}, if we have a continuous
function $u(t)$ that satisfies this, then by the Fundamental Theorem of
Calculus the $u(t)$ on the left side is differentiable and is the desired
solution of equation \eqref{simple0}.

\smallskip
For our initial approximation let $u_0(t)=0$ (this particular choice is
not very important) and recursively define
\begin{equation}\label{simple2}
u_{k+1}(t)=\alpha+ \int_0^t[a(s)u_k(s) + f(s)]\,ds, \qquad k=0,1,\ldots .
\end{equation}
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We will show that the $u_k$ converge uniformly to the desired solution
of equation \eqref{simple1}. While the optimal version is to do this on
the whole interval $[0,\,b]$, it is simpler (and, in many ways more
illuminating) to work assuming $t$ is in the smaller interval $[0,\,\beta]$,
where $\beta< \min(b,\, 1/\norm{a})$.  Note that since $u_k(t)$ is
continuous in the interval $[0,\,\beta]$, so is $u_{k+1}(t)$. The
uniform norms $\norm{u}$, we use for the $u_j$ will now be   as in
equation, except on this smaller interval $[0,\,\beta]$.  Subtracting
we obtain
\begin{equation}\label{simple3}
\begin{aligned}
u_{k+1}(t) - u_k(t) =& \int_0^t a(s)[u_k(s) - u_{k-1}(s)]\,ds\\
\le& \norm{a}\norm{u_k - u_{k-1}}\beta \le c\norm{u_k - u_{k-1}},
\end{aligned}
\end{equation}
where $c:=\norm{a}\beta<1$. Because the right side is independent of
$t$, the sequence of successive approximations is {\it contracting}
\begin{equation}\label{contract}
\norm{u_{k+1} - u_k}\le c\norm{u_k - u_{k-1}}.
\end{equation}
Using this inequality repeatedly we deduce that
\begin{equation}\label{contract2}
\norm{u_{k+1} - u_k}\le c\norm{u_k - u_{k-1}}\le \cdots \le c^k\norm{u_1 -
u_0}.
\end{equation}
Since $0<c<1$, the series $\displaystyle\sum c^k\norm{u_1 - u_0}$
converges.  Thus by the Weierstrass M-test the series
$\sum \abs{u_{k+1}(t)- u_k(t)}$ of continuous functions converges
absolutely and uniformly in the interval $[0,\,\beta]$ to some
continuous function.  But
\[
\begin{aligned}
\sum_{k=0}^N [u_{k+1}(t)- u_k(t)]
=& [u_{N+1}(t) -u_N(t)] +[u_N(t)-u_{N-1}(t)]+\cdots +[u_1(t)-u_0(t)]\\
=& u_{N+1}(t) -u_0(t)= u_{N+1}(t)
\end{aligned}
\]
Consequently the sequence of continuous functions $u_N(t)$ converges
uniformly in the interval $[0,\,\beta]$ to some continuous function $u(t)$.
We use this to let $k\to\infty$ in equation \eqref{simple2} and find
that $u(t)$ is the desired solution of equation \eqref{simple1}.  In
this last step we used the uniform convergence to interchange limit and
integral in equation \eqref{simple2}.

\smallskip
This completes the existence proof for the special case of equation
\eqref{simple0}.

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We now repeat the proof for the general system equation \eqref{int1},
using, where needed, standard facts about vectors and matrices from the
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Appendix at the end of these notes.  It is remarkable that the only
changes needed are changes in notation.
\smallskip

Just as in equation \eqref{simple2} we let
$\vU_0(t)=\vec{0}$ and recursively define
\[
\vU_{k+1}(t)=\vec{\alpha} + \int_0^t\left[A(s)\vU_k(s) +
\vec{F}(s)\right]\,ds,\qquad k=0,1,\ldots .
\]
Given the continuous $\vU_k(t)$ this defines the next approximation,
$\vU_{k+1}(t)$.   We will show that the $\vU_k$ converge uniformly to
the desired solution of equation \eqref{int1} in the smaller interval
$[0,\,\beta]$, where $\beta=\min(b,\,1/\norm{A}$.  Now
\[
\vU_{k+1}(t)- \vU_k(t) = \int_0^t A(s)\left[\vU_k(s) - \vU_{k-1}(s)\right]\,ds
\]
To estimate the right hand side we use the inequalities \eqref{int-vec}
and \eqref{norm3} from the Appendix
\begin{equation}\label{int2}
\begin{aligned}
\abs{\vU_{k+1}(t)- \vU_k(t)}
\le& \norm{A}\int_0^t\left|\vU_k(s) - \vU_{k-1}(s)\right|\,ds\\
\le& \norm{A}\norm{\vU_k - \vU_{k-1}}\beta=c\norm{\vU_k - \vU_{k-1}}
\end{aligned},
\end{equation}
where $c=\norm{A}\beta<1$.

\smallskip
The remainder of the proof goes exactly as in the previous special
case and proves the existence of a solution to equation \eqref{int1}
and hence our differential equation \eqref{ode1}.

\medskip
{\bf Uniqueness }
There are several ways to prove the uniqueness of the solution of the
initial value problem \eqref{ode1}. None of them are difficult.  We
work in the interval $[0,\,\beta]$ defined above. Say
$\vec{U}(t)$ and $\vec{V}(t)$ are both solutions.  Let
$\vec{W}(t):= \vec{U}(t) -\vec{V}(t)$.  Then $\vec{W}'=A\vec{W}$ with
$\vec{W}(0)=0$.  We want to show that $W(t)\equiv 0$.

Just as in equation \eqref{int1}
\[
\vec{W}(t)=\int_0^t A(t)\vec{W}(t)\,dt.
\]
Thus, similar to the computation in \eqref{int2}
\[
\abs{\vec{W}(t)}\le \norm{A}\norm{\vec{W}}\beta\ =\ c\norm{\vec{W}}.
\]
Because thr right-hand side is independent of $t\in[0.\,\beta]$,
\[
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\norm{\vec{W}}\le c\norm{\vec{W}}.
\]
Because $0<c<1$, this implies that $\norm{\vec{W}}=0$,  Thus
$\vec{U}(t)=\vec{V}(t)$ for $t\in [0.\,\beta]$.

\bigskip
{\sc Remark: }  There is a useful conceptual way to think of the
proof.  If $v(t)$ is a continuous function, define the map
$T:C([0,\,b)\to C([0,\,b])$ by the rule
\[
T(u)(t): =\alpha +\int_0^t [a(s)u(s) + f(s)]\,ds.
\]
Then equation \eqref{simple1} says the the solution we are seeking
satisfies $u=T(u)$.  In other words $u$ is a {\it fixed point} of the
map $T$.  The solution to many questions can be usefully attacked by
viewing them as seeking a fixed point of some map.



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\begin{center}
{\large \bf Appendix on Norms of Vectors and Matrices}
\end{center}

This is a  review of a few items concerning vectors and matrices.
Let $\vu:=(u_1,u_2,\ldots,u_n)$ and $\vv:=(v_1,\ldots,v_n)$ be points
(vectors) in $\R^n$.  Their \emph{inner product} (also called their
\emph{dot product} is defined as
\[
\ip{\vu}{\vv}=\ip{\vu}{\vv}=u_1v_1+u_2v_2+\cdots+u_nv_n.
\]
In particular, the Euclidean length
\[
\abs{\vu}^2=\ip{\vu}{\vu}=u_1^2 + u_2^2 +\cdots+u_n^2.
\]
This gives the useful formula
\begin{equation}\label{ip2}
\begin{aligned}
\abs{\vu-\vv}^2=&\ip{\vu-\vv}{\vu-\vv}
=\ip{\vu}{\vu} -2\ip{\vu}{\vv} + \ip{\vv}{\vv}\\
=&\abs{\vu}^2 -2\ip{\vu}{\vv} +\abs{\vv}^2
\end{aligned}
\end{equation}
For vectors in the plane, $\R^2$, the inner product is interpreted
geometrically as
\[
\ip{\vu}{\vv}=\abs{\vu}\abs{\vv}\cos\theta,
\]
where $\theta$ is the angle between $\vu$ and $\vv$.  Since
$\abs{\cos{\theta}}\le 1$, this implies the {\it Cauchy inequality}
\begin{equation}\label{Cauchy1}
\abs{\ip{\vu}{\vv}} \le \abs{\vu}\abs{\vv}.
```

```latex
\end{equation}
The following is a direct analytic proof the Cauchy inequality in $\R^n$
without geometric considerations -- which gave us valuable insight.  We
begin by noting that using the inner product, from equation \eqref{ip2}
for any real number $t$
\begin{equation}\label{Cauchy2}
0 \le \abs{\vu -t\vv}^2 =\abs{\vu}^2 -2t\ip{\vu}{\vv} +
t^2\abs{\vv}^2.
\end{equation}
Pick $t$ to that the right side is as small as possible (so take the
derivative with respect to $t$).  We find $t= \ip{\vu}{\vv}/
\abs{\vv}^2$.  Substituting this in inequality \eqref{Cauchy2} gives
Cauchy's inequality \eqref{Cauchy1}.

\bigskip

Next we investigate a standard system of linear equations $A\vu=\vv$
where $A=(a_{ij})$ is an $n\times n$ matrix:
\[
\begin{aligned}
a_{11}u_1+a_{12}u_2 + \cdots + a_{1n}u_n=&v_1 \\
a_{21}u_1+a_{22}u_2 + \cdots + a_{2n}u_n=&v_2 \\
\vdots\qquad \qquad\vdots\qquad \qquad&\vdots\\
a_{n1}u_1+a_{n2}u_2 + \cdots + a_{nn}u_n=&v_n
\end{aligned}
\]
Then
\[
\begin{aligned}
\abs{A\vu}^2=\abs{\vv}^2=&v_1^2+\cdots+v_n^2\\
 =& (a_{11}u_1 + \cdots + a_{1n}u_n)^2 +\cdots +
(a_{n1}u_1 + \cdots + a_{nn}u_n)^2.
\end{aligned}
\]
Applying the Cauchy inequality to each of the terms on the last
line above we find that
\begin{equation*}
\begin{aligned}
\abs{A\vu}^2 \le& \biggl(\sum_{j=1}^na_{1j}^2\biggr)\abs{\vu}^2 +\cdots
+ \biggl(\sum_{j=1}^na_{nj}^2\biggr)\abs{\vu}^2\\
=&\biggl(\sum_{i,j=1}^n a_{ij}^2\biggr)\abs{\vu}^2\\
=& \abs{A}^2\abs{\vu}^2
\end{aligned},
\end{equation*}
where we defined $\abs{A}^2=\sum_{i,j=1}^n a_{ij}^2$ (this definition
of $\abs{A}$ is often called the {\it Frobenius norm} of $A$).  Thus
\begin{equation}\label{norm2}
\abs{A\vu}\le \abs{A}\abs{\vu}.
\end{equation}

\medskip
If the elements of matrix $A=a_{ij}(t)$ and the vector
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$\vu=(v_1(t),\ldots,v_n(t))$ are continuous functions of $t$ for $t$ in
some interval $J\subset\R$, we measure the size of $A$ and
$\vu$ over the whole interval $J$ as follows:
\[
\norm{A}_J:=\sup_{t\in J}\abs{A(t)}\qquad \text{and}\qquad
\norm{\vu}_J:=\sup_{t\in J}\abs{\vu(t)}.
\]
The inequality \eqref{norm2} thus implies
\begin{equation}\label{norm3}
\norm{A\vu}_J\le \norm{A}_J\norm{\vu}_J.
\end{equation}

\bigskip
There is one more fact we will need about integrating a continuous
vector-valued function $\vv(t)$ on an interval $[a,\,b]$.  It is the
inequality
\begin{equation}\label{int-vec}
\biggl|\int_a^b \vv(s)\,ds\biggr| \le  \int_a^b\abs{\vv(s)}\,ds
\end{equation}
To prove this, note that we define the integral of a vector-valued
function $\vv(t)=(u_1(t),\ldots,u_n(t))$ as integrating each
component separately. Using this and the Cauchy inequality, we find
that for any constant vector $\vec{V}$,
\[
 \ip{\vec{V}}{\int_a^b\vv(s)}\,ds= \int_a^b\ip{\vec{V}}{\vv(s)}\,ds \le
\abs{\vec{V}}\int_a^b\abs{\vv(s)}\,ds.
\]
In particular, if we let let $\vec{V}$ be the constant vector
$\vec{V}=\int_a^b \vv(s)\,ds$,  then
\[
\abs{\vec{V}}^2 =\ip{\vec{V}}{\vec{V}}=\ip{\vec{V}}{\int_a^b\vv(s)}\,ds
\le \abs{\vec{V}}\int_a^b\abs{\vv(s)}\,ds.
\]
After canceling $\abs{\vec{V}}$ from both sides this is exactly the
desired inequality \eqref{int-vec}.

\end{document}
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