Math 202, Fall 2018 Jerry L. Kazdan

Problem Set 9

DUE: In class Thursday, Nov. 15. *Late papers will be accepted until 1:00 PM Wednesday.*

REMARK: Please read Chapter 17 on Integration.

PROBLEMS

1. [#16.34] Let f be differentiable on the interval $[a,\, b]$, and suppose that both f and f' are positive. Prove that the function $g := f/(1+f)$ is bounded and increasing.

2. [#16.49 Suppose that f and g are convex (not necessarily differentiable) and $c \in \mathbb{R}$. Which of the three functions $f+g$, $c \cdot f$, and $f \cdot g$ must be convex? (Give proofs or counterexamples.

3. a) [#16.52] Which polynomials of odd degree are convex on $\mathbb{R}$?

   b) [#16.52] Characterize the polynomials of degree four that are convex on $\mathbb{R}$ by giving a necessary and sufficient condition on the coefficients.

4. Let $f_n(x) := x^n(1-x)$. Prove that $f_n \to 0$ uniformly on $[0,\, 1]$.

5. [#16.61] Let $f_n(x) := x^2/(x^2+n^2)$.

   a) Prove that f_n converges pointwise to 0 on $\mathbb{R}$.

   b) Prove that f_n does not converge uniformly to 0 on $\mathbb{R}$.

   c) Does f_n converge uniformly to 0 on the interal $[0,\, 100]$? Justify your assertion.

6. Use the definition of the integral as a Riemann sum to compute $\int_0^b x^3\, dx$. You may use that $\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2}\right)^2$.

7. Use the definition of the integral as a Riemann sum to compute $\int_0^b \cos x\, dx$. You will need the formula for $\cos\theta + \cos 2\theta + \cos 3\theta + \cdots \cos n\theta$.

8. [#17.15] Let f be continuous on the interval $[a,b]$ and assume that $f(x) \ge 0$ for all $a \le x \le b$. Use the definition of the integral as a Riemann sum to show that if $\int_a^b f(x)\, dx = 0$, then $f(x) = 0$ everywhere. [You will need to use that since f is continuous, if it is positive at some point, then it is positive in some interval containing the point.]

Bonus Problems

[Please give your solutions directly to Professor Kazdan]

1-B a) Let $p(x) := x^3 + cx + d$, where c, and d are real. Under what conditions on c and d does this has three distinct real roots? [ANSWER: $c < 0$ and $d^2 < -4c^3/27$].

b) For a general cubic polynomial, $p(x) := ax^3 + bx^2 + cx + d \quad (a \neq 0)$, let α be the point where $p''(x) = 0$ and make the change of variables $x = t + \alpha$. Compute p as a polynomial as a function of t. Note that here the coefficient of t^2 is zero.

c) Use this observation, generalize part a) to the real polynomial $p(x) := ax^3 + bx^2 + cx + d \quad (a \neq 0)$.

[Last revised: November 9, 2018]