Math 202, Fall 2018 Jerry L. Kazdan

Problem Set 8

DUE: In class Thursday, Nov. 8. *Late papers will be accepted until 1:00 PM Friday.*

REMARK: Please re-read Chapter 16 on Differentiation.

PROBLEMS

1. a) Let x_j, $j = 1, 2, \ldots$ be a sequence of points in $\mathbb{R}$. If $|x_{j+1} - x_j| \le \dfrac{1}{j^4}$, show that these points converge. [HINT: Cauchy sequence]

   b) A generalization. Let $\{x_j\}$ be a sequence of points in $\mathbb{R}$ with the property that

   $$\sum_j |x_{j+1} - x_j| < \infty.$$

   Prove that the sequence $\{x_j\}$ converges.

   Give an example of a convergent sequence that does not have this property.

2. This problem concerns the continuity of $f(x) = \frac{1}{x}$ at a point $x = a > 0$. Let $\epsilon = 1$.

   a) At the point $a = 1/10$, find a $\delta > 0$ so that if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$.

   b) At the point $a = 1/1000$, find a $\delta > 0$ so that if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$.

3. Show that $f(x) := 1/x$ is uniformly continuous on the interval $x \ge 1$.

4. Let $f(x)$ be differentiable at every point of the open interval $a < x < b$ (possibly unbounded).

   a) If the derivative is bounded, say $|f'(x)| \le M$, in this interval, show that f is uniformly continuous in the interval.

   b) If the derivative is **not** bounded in this interval, show that f is **not** uniformly continuous in the interval.

   c) Apply these to the functions x^2 and $1/x$ on the interval $x \ge 1$.

5. Let c_1, c_2, $\ldots$, c_n be any real numbers and let

   $$Q(x) := (x - c_1)^2 + (x - c_2)^2 + \cdots + (x - c_n)^2.$$

   Find the value of x that minimizes Q and prove that it gives a minimum.

6. Let $u(x)$ and $v(x)$ both be solutions of the same differential equation $y'' + cy = 0$, where c is a constant. Show that the function $\varphi := uv' - u'v$ is a constant. [Note: This problem is *not* asking you to solve this differential equation.]

   [REMARK: In the special case $c = 1$, this implies that $\sin^2 x + \cos^2 x = 1$.]

7. Prove that two differentiable functions on the interval $(0, 1)$ have the same derivative if and only if they differ by a constant.

8. Show that $e^x \ge 1 + x$ for all (real) x. [SUGGESTION: Use $(e^x)'' = e^x > 0$ so e^x is convex – and hence at every point the graph lies above its tangent line].

9. If for some constant γ the differentiable function $v(x)$ satisfies $v' - \gamma v \le 0$, show that $v(x) \le v(0)e^{\gamma x}$ for all $x \ge 0$. [HINT: Consider $g(x) := e^{-\gamma x} v(x)$.]

10. Say a function $f(x)$ has the property that there is a constant $\alpha > 0$ and a constant $C > 0$ so that

$$|f(x_2) - f(x_1)| \le C|x_2 - x_1|^{1+\alpha} \qquad \text{for all real} \quad x_1, x_2.$$

Show that $f(x) \equiv$ constant.

Also give an example of a *non-constant* function that does satisfy this with $\alpha = 0$.

11. Let $f(x) : \mathbb{R} \to \mathbb{R}$ be a bounded smooth function for $-\infty < x < \infty$. Show that $f''(c) = 0$ for at least one point c. Thus, f has at least one inflection point.

12. The point of this problem is to get *all* of the properties of e^x by defining it as the solution of the differential equation

$$E'(x) = E(x) \quad \text{with initial condition} \quad E(0) = 1. \tag{1}$$

Thus, for this problem, don't use any properties of $E(x)$ other than this equation.

a) Show that $E(x)E(-x) = 1$. In particular. $E(x) \ne 0$ for any x and $E(-x) = 1/E(x)$. Note that since $E(0) > 0$, then $E(x) > 0$ for all x. [SUGGESTION: Let $F(x) := E(-x)$, show that $F'(x) = -F(x)$ and then take the derivative of $w(x) := F(x)E(x)$.]

b) Show that

$$E(x + a) = E(x)E(a). \tag{2}$$

[SUGGESTION: Let $u(x) := E(-x)E(x + a)$ and compute $u'(x)$].

c) Use (2) to show that $E(kx) = E(x)^k$ for any integer k. The three special cases $x = 1$, $x = 1/k$, and $x = n$ give $E(n/k) = E(1)^{n/k}$. Since every real x is the limit of rationals, by the continuity of $E(x)$ we find $E(x) = E(1)^x$. *Defining* $e := E(1)$, we obtain $E(x) = e^x$ for all real x.

Bonus Problems

[Please give your solutions directly to Professor Kazdan]

1-B This problem concerns a classical example of a function $f(x) \not\equiv 0$ that has derivatives of all order and all of whose derivatives are zero at the origin are zero. Thus its Taylor polynomials at the origin are all zero. While at first this seems like pathology, this function is quite useful.

a) Let $x > 0$. Show that $\lim_{x\to 0} \frac{1}{x^k} e^{-1/x} = 0$ for any integer k.

b) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{for } x > 0, \\ 0 & \text{for } x \le 0, \end{cases}.$$

Show that f is smooth, that is, $f(x)$ and all of its derivatives exist and are continuous for all real x. Sketch the graph.

c) Show that each of the following are smooth and sketch their graphs:

$$g(x) = f(x)f(1-x), \qquad h(x) = \frac{f(x)}{f(x) + f(1-x)},$$
$$k(x) = h(x)h(4-x), \qquad K(x) = k(x+2), \qquad H(x) = K(4x)$$

2-B Say a function $u(x)$ satisfies the differential equation

$$u'' + b(x)u' + c(x)u = 0 \tag{3}$$

on the interval $[0, A]$ and that the coefficients $b(x)$ and $c(x)$ are both bounded, say $|b(x)| \le M$ and $|c(x)| \le M$ (if the coefficients are continuous, this is always true for some M).

a) Define $E(x) := \frac{1}{2}(u'^2 + u^2)$. Show that for some constant γ (depending on M) we have $E'(x) \le \gamma E(x)$. [SUGGESTION; use the inequality $2ab \le a^2 + b^2$.]

b) Use Problem 2(c) above to show that $E(x) \le e^{\gamma x} E(0)$ for all $x \in [0, A]$.

c) In particular, if $u(0) = 0$ and $u'(0) = 0$, show that $E(x) = 0$ and hence $u(x) = 0$ for all $x \in [0, A]$. In other words, if $u'' + b(x)u' + c(x)u = 0$ on the interval $[0, A]$ and that the functions $b(x)$ and $c(x)$ are both bounded, and if $u(0) = 0$ and $u'(0) - 0$, then the only possibility is that $u(x) \equiv 0$ for all $x \ge 0$.

d) Use thuis to prove the *uniqueness theorem*: if $v(x)$ and $w(x)$ both satisfy equation (3) and have the same initial conditions, $v(0) = w(0)$ and $v'(0) = w'(0)$, then $v(x) \equiv w(x)$ in the interval $[0, A]$.

3-B [INTERPOLATION] Say $f(x)$ is a smooth function on the interval $[a, b]$ and you approximate it by the chord

$$p(x) := f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

that passes through the end points $(a, f(a))$ and $(b, f(b))$. We seek an estimate for the error, $f(x) - p(x)$.

Fix this specific point x ($x \neq a$ and $x \neq b$) and let

$$E(t) := f(t) - [p(t) + K(t-a)(t-b)],$$

where the constant K is chosen so that $E(x) = 0$. Show that $K = \frac{1}{2}f''(c)$ for some point $c \in (a, b)$ and conclude that

$$\begin{aligned} f(x) =& p(x) + \frac{1}{2}f''(c)(x-a)(x-b) \\ =& f(a) + \frac{f(b)-f(a)}{b-a}(x-a) + \frac{1}{2}f''(c)(x-a)(x-b) \end{aligned}$$

This also shows that if $f \geq 0$ then for x in the interval $[a, b]$, the graph of $f(x)$ lies below the chord joining the points $(a, f(a))$ and $(b, f(b))$.

[Last revised: October 27, 2018]