Math 202, Fall 2018 Jerry L. Kazdan

Problem Set 7

DUE: In class Thursday, Oct. 25. *Late papers will be accepted until 1:00 PM Friday.*

REMARKS:
Please read Chapter 16, p. 307-324 on Differentiation and also the class notes
`http://www.math.upenn.edu/~kazdan/202F13/notes/sqrt-Newton.pdf` on Newton's Method.

PROBLEMS

1. [# 16.1] For $x \neq 0$, determine $\lim_{h\to 0} \frac{1}{h}\left(\frac{1}{(x+h)^2} - \frac{1}{x^2}\right)$. [REMARK: Think before computing!]

2. For each part, determine if the statement is True or False. It True, give a proof; if False, give a counterexample.
   a) [# 16.6] There is a function f so that $f(x+h) = f(x) + h$ for all real x, h.
   b) [# 16.7] There is a function f so that $f(x+h) = f(x) + h^2$ for all real x, h.

3. [# 16.19] *Sufficient conditions for differentiability at a point.*
   a) Suppose that $|f(x)| \le x^2 + x^4$ for all x. Prove that $f'(0)$ exists.
   b) Suppose that $|f(x)| \le g(x)$ where $g(x) \ge 0$ for all x and $g'(0) = g(0) = 0$. Prove that $f'(0)$ exists.
   c) Suppose that $g(x)$ is a bounded function and that $f(x) := (x-a)^2 g(x)$ for all x. Prove that $f'(a)$ exists.

4. Use the definition of the derivative as the limit of a difference quotient to show that $\cos x$ is differentiable for all x. [You may use without proof that $\lim_{\theta\to 0} \sin\theta/\theta = 1$ and $\lim_{\theta\to 0}(1-\cos\theta)/\theta = 0$.]

5. Let $a_n = (n+1)^{2/5} - n^{2/5}$ for $n = 1, 2, \ldots$. Show that $\lim_{n\to\infty} a_n \to 0$. [HINT: Mean Value Theorem]

6. Show, directly from the definition, that $\sqrt{x}$ is continuous at every $x \ge 0$. Is it uniformly continuous for every $x \in [0,\infty)$? Why? [You may find it useful to split $[0, \infty)$ into $[0, 1]$ and $[1, \infty)$.]

7. Which of the following are uniformly continuous in the set $\{x \ge 0\}$? Justify your assertions.

   a). $f(x) = 2 + 3x$ b). $g(x) = \sin 2x$ c). $h(x) = x^2$.

8. Let $f : [0, \infty) \to \mathbb{R}$ be differentiable for all $x \geq 0$. If its derivative is bounded, say $|f'(x)| \leq M$, prove that it is uniformly continuous. [HINT: Mean Value Theorem]

9. Let a smooth function $g(x)$ have the properties: $g(0) = 3$, $g(1) = 1$, and $g(4) = 7$.

   a) Show that at some point $0 < c < 4$ one has $g''(c) > 0$. Better yet, find a number $m > 0$ so that $g''(c) \geq m > 0$.

   b) Is it true that g'' must be positive at at least one point in the interval $0 < x < 1$? Proof or counterexample.

   c) [This is the optimal version of part (a)]. Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be any three points in the plane with $x_1 < x_2 < x_3$, $y_1 > y_2$, and $y_3 > y_2$. Then there is a point $c \in (x_1, x_3)$ such that $g''(c) = m > 0$, where m is the second derivative of the (unique) quadratic polynomial passing through the three points.

10. Let $g(x)$ is a smooth function with $g(2) = 0$ and let $f(x) = x^2 g(x)$. Use the mean value theorem to show that $f''(c) = 0$ for some $0 < c < 2$.

11. a) Let $g(x) := x^3(1 - x)$. Use the mean value theorem to show that $g'''(c) = 0$ for some $0 < c < 1$.

    b) Let $h(x) := x^3(1 - x)^3$. Show that $h'''(x)$ has exactly three distinct roots in the interval $0 < x < 1$.

    c) Let $p(x) := \left(\frac{d}{dx}\right)^4 (1 - x^2)^4$. Show that p is a polynomial of degree 4 and that it has 4 real distinct zeroes, all lying in the interval $-1 < x < 1$.

12. If $b \geq 0$, show that for every real c the equation $x^5 + bx + c = 0$ has exactly one real root.

13. [#16.31] Let $f(x)$ be a differentiable function for all real x with the property that $f'(x) < 1$ for all x. Show has at most one *fixed point*, that is, at most one point p where $f(p) = p$.

14. Let $f(x)$ be a differentiable function for all real x with the property that $|f'(x)| < 1/2$ for all x. Define the sequence x_k by the rule $x_1 = 1$ and $x_{k+1} = f(x_k)$ for $k = 1, 2, \ldots$. Show that the x_k converge to a point p and that $f(p) = p$, so p is a fixed point of f. [SUGGESTION: Use the mean value theorem to show that

$$|x_{k+1} - x_k| \leq \tfrac{1}{2}|x_k - x_{k-1}|$$

and then use work we did earlier (contracting sequences) to conclude that the x_k is a Cauchy sequence etc.

Bonus Problems

[Please give your solutions directly to Professor Kazdan]

1-B In last week's homework you showed that if $f(x)$ is a continuous real-valued function with the property

$$f(x+y) = f(x) + f(y)$$

for all real x, y, then $f(x) = cx$ where $c := f(1)$.

Give a very short proof if you make the stronger assumption that f is differentiable.

2-B [COBWEB MODEL IN ECONOMICS] Read the Wikipedia article

`https://en.wikipedia.org/wiki/Cobweb_model`

on supply and demand in economics. Prove the assertions made on the convergent and divergent cases.

You may assume the supply and demand curves are "nice" but clearly state any other assumptions you make. [To get started, first do the special case where the supply and demand curves are both straight lines.]

3-B Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous. Prove that there are constants a, b such that $|f(x)| \le a + b|x|$ for all x.

[Last revised: October 19, 2018]