Math 202, Fall 2018 Jerry L. Kazdan

Problem Set 6

DUE: In class Thursday, Oct. 18. *Late papers will be accepted until 1:00 PM Friday.*

REMARKS:
Please re-read Chapter 15 on Continuity and read Chapter 16, pages 307-317 on Differentiation.

PROBLEMS

1. In class, if $z = x + iy$ we defined e^z by a power series and observed that, $e^{iz} = \cos z + i \sin z$. In particular, for real x, $e^{ix} = \cos x + i \sin x$. From the power series one cam also show that for any complex z and w the usual formula $e^{z+w} = e^z e^w$ remains valid. Use the observation that
$$1 + \cos x + \cos 2x + \cdots + \cos nx = \text{Real part of } \{1 + e^{ix} + e^{2ix} + \cdots + e^{nix}\}$$
and that the right hand side is a geometric series to find a formula for $1 + \cos x + \cos 2x + \cdots + \cos nx$. [Assume x is not a multiple of 2π]. Your resulting formula should not have any complex numbers.

2. [#15.3] [T/F] If $f : \mathbb{R} \to \mathbb{R}$ is continuous everywhere and $f(x) = 0$ for all rational numbers x, then $f(x) = 0$ for all real x.

3. [#15.5] [T/F] The function $f(x) := |x|^3$ is continuous for all $x \in \mathbb{R}$.

4. [#15.7] [T/F] Let f, g, and h be continuous on the interval $[0, 2]$. If $f(0) < g(0) < h(0)$ and $f(2) > g(2) > h(2)$, then there exists some $c \in [0, 2]$ such that $f(c) = g(c) = h(c)$.

5. [#15.10][T/F]
   a) If f is continuous on $\mathbb{R}$, then f is bounded.
   b) If f is continuous on $[0, 1]$, then f is bounded.
   c) If f is continuous on $\mathbb{R}$ and is bounded, then f attains its supremum.

6. [#15.12] Construct a function f with the property that there are sequences a_n and b_n converging to zero such that $f(a_n)$ converges to zero but $f(b_n)$ is unbounded.

   Does there exist such a function f that is continuous at $x = 0$?

7. [#15.15] Let $f(x) := x^2 + 4x$. Clearly $\lim_{x\to 0} f(x) = 0$. Assuming that $0 < \epsilon < 4$, how small must δ be so that $|x| < \delta$ implies that $|f(x)| < \epsilon$? Express δ as a function of ϵ.

8. [#15.17] Let $f(a,n) := (1+a)^n$, where a and n are positive.

   a) For constant a, how does $f(a,n)$ behave as $n \to \infty$? For constant n, how does $f(a,n)$ behave as $a \to 0$?

   b) Let $L \geq 1$ be a given real number. Prove that there exists a sequence $a_n \to 0$ and $f(a_n,\, n) \to L$ as $n \to \infty$. In other words, depending on the choice of a_n, f may approach any value.

9. [#15.21] Prove that there exists $x \in [1,\, 2]$ such that $x^5 + 2x + 5 = x^4 + 10$.

10. Given any real number $c > 0$, prove there is an $x > 0$ such that $x^{17} + 8x^2 = c$.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous (real-valued) function that is continuous at $x = a$. If $f(a) > 0$, show there is an interval $J := \{x \in \mathbb{R} \,|\, |x - a| < \delta\}$ so that if $x \in J$, then $f(x) > f(a)/2$.

12. [#15.24] Prove that any (real) polynomial whose degree is odd must have at least one real root.

13. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property:

$$f(x+y) = f(x) + f(y) \quad \text{for all real } x \text{ and } y$$

   and let $c := f(1)$

   a) Show that $f(0) = 0$.

   b) Show that $f(-x) = -f(x)$ for all real x.

   c) If k is a positive integer show that $f(kx) = kf(x)$ for all x.

   d) If k and n are positive integers, show that $f(k) = kf(1) = kc$ and $f(1/n) = c/n$.

   e) If $x = p/q$ is a rational number, show that $f(x) = cx$.

   f) If x is a real number, show that $f(x) = cx$.

14. Say $g : \mathbb{R} \to \mathbb{R}$ is a continuous function with the property:

$$g(x+y) = g(x)g(y) \quad \text{for all real } x \text{ and } y.$$

   What can you conclude about g?

Bonus Problem

[Please give your solutions directly to Professor Kazdan]

1-B The number e is defined as

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots .$$

Prove that e is *not* a rational number by the following steps.

a) Show that $2 < e < 3$. So e is definitely not an integer.

b) By contradiction, say $e = \frac{p}{q}$, where p and q are positive integers with $q \geq 2$. Show that

$$e\,q! = N + \frac{c}{q+1},$$

where N is an integer and $0 < c < e$. Thus, conclude that $\frac{c}{q+1}$ must be an integer.

c) Then show that this contradicts $e < 3$ and $q + 1 \geq 3$.

[Last revised: October 13, 2018]