Problem Set 4

DUE: In class Thursday, Sept. 27 *Late papers will be accepted until 1:00 PM* Friday.

Please reread Chapter 14 and Chapter 18

As always, in a number of the problems, the issue is to find the key idea and state it clearly. And then to have a well organized explanation that is easy to read and understand.

Exam 1 will be held in class on Tuesday Oct. 2. Closed book but you may use one 3" $\times$5" card with handwritten notes on both sides.

PROBLEMS

[Note that you are *not* required to submit your homework in TEX or LATEX.]

1. [#13.27] Let $a_n = \sqrt{n^2+n} - n$. Show that it converges and compute the limit.

2. A real (or complex) sequence x_n is called *contracting* if for some constant $0 < c < 1$ (such as $c = \frac{1}{2}$) it has the property that for all $n = 1, 2, 3, \ldots$

$$|x_{n+1} - x_n| \le c|x_n - x_{n-1}|.$$

The point of this problem is to show that a contracting sequence converges.

a) Show that $|x_{n+1} - x_n| \le c^n|x_1 - x_0|$ for all n.

b) Use $x_{n+1} - x_0 = (x_{n+1} - x_n) + (x_n - x_{n-1}) + \cdots + (x_1 - x_0)$ to show that

$$|x_{n+1} - x_0| \le \left(c^n + c^{n-1} + \cdots + c + 1\right)|x_1 - x_0|$$

c) More generally, if $n > k$ show that

$$\begin{aligned} |x_{n+1} - x_k| \le & \left(c^n + c^{n-1} + \cdots + c^k\right)|x_1 - x_0| \\ =& c^k\left(\frac{1 - c^{n-k+1}}{1-c}\right)|x_1 - x_0| < c^k\frac{|x_1 - x_0|}{1-c}. \end{aligned}$$

d) Show that the x_n are a Cauchy sequence — and hence converge.

3. Let $a_1 = 1$ and define the a_n, $n \ge 2$, recursively by the rule $a_{n+1} := \sqrt{1 + a_n}$. Prove this sequence converges. Then find the value of the limit.

4. If $0 < p < 1$, show that the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges. [We did the case $p = 1/2$ in class.]

5. [#14.50] Determine whether $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ converges.

6. Consider the *alternating series* $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ and let S_n be the sum of the first n terms.

   a) Show that the even terms, S_2, S_4,... S_{2n} are increasing and the odd terms, S_1, S_3, ..., S_{2n+1}, are decreasing.

   b) Show that $S_{2n+1} - S_{2n} = 1/(2n+1) > 0$.

   c) Show that the even terms S_{2n} converge to some number α and the odd terms S_{2n+} converge to some number β.

   d) Show that $\alpha = \beta$ and conclude that the whole series converges.

7. [#14.60] Which of the following series converge and which diverge? Why?

$$a).\ \sum_{n=1}^{\infty} \frac{2n^2+15n+2}{n^4+3n+1}, \qquad b),\ \sum_{n=1}^{\infty} \frac{2n^2+15n+2}{n^3+3n+1}, \qquad c).\ \sum_{n=1}^{\infty} \frac{3+5n+n^2}{2^n}.$$

8. a) Let $\mathbb{N}_k$ be the set of the first k positive integers $1, \ldots, k$ and L the set of positive integers *not* divisible by 2. Define the sequence

$$S_k := \frac{\text{number of integers in the set } L \cap \mathbb{N}_k}{k}.$$

   Compute $\lim_{k\to\infty} S_k$.

   b) Let M the set of positive integers *not* divisible by 2 or 3. Define the sequence

$$T_k := \frac{\text{number of integers in the set } M \cap \mathbb{N}_k}{k}.$$

   Compute $\lim_{k\to\infty} T_k$.

9. Let a_n be a real sequence that converges to L and b_n be a real sequence that converges to M. Define a new sequence $c_n := \max(a_n, b_n)$. Either prove that this sequence converges or give a counterexample.

[Last revised: September 21, 2018]