Problem Set 3

DUE: In class Thursday, Sept. 20. *Late papers will be accepted until 1:00 PM Friday.*

Lots of problems this week. Fortunately a number of them are short – but don't wait until Wednesday night!

Enjoy the weekend ... and have fun with the problems.

Jerry Kazdan

1. Please reread Chapter 14, pages 271 - 279.
2. Please read Chapter 18, Sections 18.1 - 18.7.
3. Note that Exam 1 will be on Tuesday, October 2 in class from 10:30–11:50. Closed book but you may use one 3×5 card with notes on both sides. It will cover Chapters 1, 13, 14 (pages 271-279), and 18 (Sections 18.1 - 18.7) from the book.

PROBLEMS

1. A tennis ball is dropped from a height H. After each bounce it returns to two-thirds of its height on the previous bounce. How far does the ball travel until it is at rest on the floor?

2. [# 13.24] let $f,\ g : \mathbb{R} \to \mathbb{R}$ be bounded functions such that $f(x) \le g(x)$ for all x. Let F denote the image of f and G the image of g. Give examples (with pictures) of pairs of such functions with:

   a). $\sup(F) < \inf(G)$ b). $\sup(F) = \inf(G)$ c). $\sup(F) > \inf(G)$

3. [# 13.30] Let $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$. Show that $\lim_{n\to\infty} x_n$ exists. [REMARK: In fact, the limit equals $\ln 2$ but that is not needed for this exercise.]

4. [# 13.32] *Nested Interval Property.* Let $\{I_n \subset \mathbb{R}\}$ be a sequence of closed (non-empty) intervals with I_n having length d_n such that $I_{n+1} \subseteq I_n$ and $d_n \to 0$. The Nested Interval Property states that for such a sequence there is exactly one point that belongs to all of the I_n.

   a) Show that our Completeness Axiom implies the Nested Interval Property.

   b) Show that the Nested Interval Property implies our Completeness Axiom.

5. [#14.22] If $c > 0$, show that $\lim_{n\to\infty} c^{\frac{1}{n}} = 1$. [SUGGESTION: The case $c = 1$ is clear. If $c > 1$, let $x_n := c^{\frac{1}{n}} - 1$ and show that $x_n \to 0$. Note $x_n > 0$ so use $c = (1 + x_n)^n \ge 1 + nx_n$. If $0 < c < 1$, take reciprocals.]

6. [#14.2] For each condition below, give an example of an *unbounded* sequence such that $a_{n+1} - a_n > 0$ for all $n \in \mathbb{N}$ and the specified condition holds.

   a) $\lim (a_{n+1} - a_n) = 0$.

   b) $\lim (a_{n+1} - a_n)$ does not exist.

   c) $\lim (a_{n+1} - a_n) = L$, where $L > 0$.

7. Suppose that $x_0 = c$ for some real c and $x_{n+1} = \sqrt{1 + x_n^2}$ for all $n \in \mathbb{N}$. For which c does x_n converge? Why?

8. [#14.9]. Proof or counterexample. Suppose that $x_n \to L$.

   a) For all $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $|x_{n+1} - x_n| < \epsilon$.

   b) There exists $n \in \mathbb{N}$ such that for all $\epsilon > 0$, $|x_{n+1} - x_n| < \epsilon$.

   c) There exists $\epsilon > 0$ such that for all $n \in \mathbb{N}$: $|x_{n+1} - x_n| < \epsilon$.

   d) For all $n \in \mathbb{N}$ there exists $\epsilon > 0$ such that $|x_{n+1} - x_n| < \epsilon$.

9. Let s_n be a sequence of real numbers that converge to some $S > 0$. Show there is an integer $N > 0$ so that if $n \geq N$ then $s_n > S/2$.

10. [#14.14] Let a_n and $b_n \neq 0$ be real sequences. If $a_n \to L$ and $b_n \to M \neq 0$, show that $a_n/b_n \to L/M$.

    SUGGESTION: First do the special case where all the $a_n = 1$, the $b_n > 0$, and $M > 0$. [Where in your proof did you use that $M > 0$?]. Then use Theorem 14.5b to get the general case.

11. [#14.15] Let b and L be real numbers. If $b \leq L + \epsilon$ for all $\epsilon > 0$, prove that $b \leq L$.

12. [#14.18] If $a_1 = 1$ and $a_{n+1} = \sqrt{3a_n + 4}$ for $n \geq 1$, show that $a_n < 4$ for all $n \geq 1$.

13. [#14.19]. Suppose that $x_1 = 1$ and $2x_{n+1} = x_n + 3/x_n$ for $n \geq 1$. Prove that $\lim_{n\to\infty} x_n$ exists – and find the limit.

14. [#18.6] If w_1 and w_2 are distinct points in $\mathbb{C}$. give a geometric description of the set $\{z \in \mathbb{C} : \; |z - w_1| = |z - w_2|\}$.

15. [#18.7] Prove the following properties of complex conjugation for all complex numbers z and w:

$$a).\ \overline{zw} = \bar{z}\bar{w} \qquad b).\ \overline{z + w} = \bar{z} + \bar{w} \qquad c).\ |\bar{z}| = |z|$$

16. If c is a complex number with $|c| < 1$, show that $(n^2+1)c^n \to 0$. Does $n^5 c^n$ converge? If so, to what? Explain your reasoning. [SUGGESTION: Prove (and use) that a complex sequence z_n converges to zero if and only if the real sequence $|z_n|$ converges to zero.]

[Last revised: September 19, 2018]