Math 202, Fall 2018 Jerry L. Kazdan

Problem Set 2

DUE: In class Thursday, Sept. 13. *Late papers will be accepted until 1:00 PM Friday.*

REMARKS:
Lots of problems. Fortunately, most are short.

In doing the proofs, it is no longer necessary to justify each arithmetic statement as we did in the beginning of the course. Just do common sense arithmetic when needed. The it only exception is careful use of the Least Upper Bound axiom.

In a number of the problems, the issue is to find the key idea and state it clearly. And then to have a well organized explanation that is easy to read and understand.

It's more important to concentrate on this than on the kind of formal proofs (statement, reason, statement, reason, etc.) that we had at the beginning.

...and to have fun with the problems.

0. On Tuesday at the beginning of class I;ll briefly introduce you to TeX and LaTeX. **Bring your laptops**.

1. Please reread Chapter 13, pages 256 - 267.
2. Please read Chapter 14, pages 271 - 279.

Here is a clearer presentation of what we did at the end of class on Thursday.

Let a_n be a sequence of real numbers. Assume $a_n \ge 0$ and $a_n \to A$. Show that $A \ge 0$.
PROOF $a_n \to 0$ means that given any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that if $n > N$ then

$$|a_n - A| < \epsilon \qquad \text{that is,} \qquad -\epsilon < a_n - A < \epsilon.$$

The inequality $a_n - A < \epsilon$ means $a_n - \epsilon < A$. Since $a_n \ge 0$ we find $-\epsilon < a_n - \epsilon < A$ for *any* $\epsilon > 0$. Thus $A \ge 0$. Done.

PROBLEMS

1. [p. 23 #30] Let x. y, u, and v be real numbers.
   a) Prove that $(xu + yv)^2 \le (x^2 + y^2)(u^2 + v^2)$.
   b) Determine precisely when equality holds in part a).

2. [p. 24 #45] Determine if the rules below define functions from $\mathbb{R}$ to $\mathbb{R}$.
   a) $f(x) = |x - 1|$ if $x < 4$ and $f(x) = |x| - 1$ if $x > 2$.
   b) $f(x) = |x - 1|$ if $x < 2$ and $f(x) = |x| - 1$ if $x > -1$.
   c) $f(x) = ((x + 3)^2 - 9)/x$ if $x \ne 0$ and $f(x) = 6$ if $x = 0$.
   d) $f(x) = ((x + 3)^2 - 9)/x$ if $x > 0$ and $f(x) = x + 6$ if $x < 7$.
   e) $f(x) = \sqrt{x^2}$ if $x \ge 2$, $f(x) = x$ if $0 \le x \le 4$, and $f(x) = -x$ if $x < 0$.

3. [p. 24 #46] Determine the *images* of the functions $f : \mathbb{R} \to \mathbb{R}$ defined below.

   a) $f(x) = \frac{x^2}{1+x^2}$.

   b) $f(x) = \frac{x}{1+|x|}$

4. [# 13.9] Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) := \dfrac{2x-8}{x^2-8x+17}$. Then the supremum of the image of f is 1. Give a proof or counterexample.

5. Show that $\lim_{n\to\infty} \dfrac{10^n}{n!} = 0$.

6. [#13.11] Suppose the sequences a_n and b_n of real numbers both converge. For each of the following assertions give a proof or counterexample,

   a) If $\lim a_n < \lim b_n$, then there exists $N \in \mathbb{N}$ such that $n \geq N$ implies that $a_n < b_n$.

   b) If $\lim a_n \leq \lim b_n$, then there exists $N \in \mathbb{N}$ such that $n \geq N$ implies that $a_n \leq b_n$.

7. [# 13.22] For each set S below, determine whether it is bounded, and determine $\sup(S)$ and $\inf(S)$, if they exist.

   a) $S = \{x : x^2 < 5x\}$

   b) $S = \{x : 2x^2 < x^3 + x\}$

   c) $S = \{x : 4x^2 > x^3 + x\}$

8. a) Use the definition of limit to show that $\lim \dfrac{2n}{1+n} = 2$.

   b) [# 13.25] Use the definition of limit to show that $\lim \sqrt{1+n^{-1}} = 1$.

9. [# 13.34]

   a) Given any two rational numbers $r < s$, prove there is an *irrational* number c between them: $r < c < s$.

   b) Given any two real numbers $a < b$, prove there is a *rational* number r between them: $a < r < b$.

10. Consider the set $\mathcal{R}$ of rational functions $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials with real coefficients and $q(x)$ not identically zero. The function $f(x)$ has a finite value everywhere except at a finite number of points (the zeroes of $q(x)$).

a) It should be obvious that with the usual definitions of addition and multiplication, the set of rational functions is closed under addition and multiplication, that is, the sum and product of two rational functions is also a rational function. Show that $\mathcal{R}$ forms a field.

b) In $\mathcal{R}$, define the order $f \succ 0$ to mean that $f(x) > 0$ for all sufficiently large *positive* real x. Thus, if

$$f(x) = \frac{a_0 + a_1 x + \cdots + a_k x^k}{b_0 + b_1 x + \cdots + b_n x^n}$$

with $a_k \neq 0$ and $b_n \neq 0$, then for large x, the function $f(x)$ is approximately $a_k x^k / b_n x^n = (a_k/b_n) x^{k-n}$. Thus $f \succ 0$ is equivalent to the algebraic statement $\dfrac{a_k}{b_n} > 0$. [This gives an *algebraic* definition of $f \succ 0$ that avoids defining "sufficiently large".] Then $f \succ g$ is defined to mean $f - g \succ 0$.

With this order relation, show that $\mathcal{R}$ is an ordered field.

c) Which of the following rational functions are positive?

$$i).\ f(x) := \frac{2}{-3 + 2x^4}, \qquad ii).\ g(x) := \frac{2x - 3x^4 + x^7}{x^5 - x^6}, \qquad iii).\ h(x) := \frac{2 - 5x^3}{x - x^2 - x^4}.$$

d) Show that this ordered field is *non-archemedean* by exhibiting two specific rational functions f and g with the property that there is *no* integer N such that $Nf > g$.

[Last revised: September 7, 2018]