Math 202 | Exam 3 | Jerry L. Kazdan

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DIRECTIONS: Part A has 6 short questions (5 points each), Part B has 2 shorter problems (8 points each), Part C has 4 traditional problems (12 points each). 94 points total].
To receive full credit your solution should be clear and correct. Neatness counts. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3 ×5 with notes on both sides.

PART A: Six shorter problems, 5 points each [total: 30 points]

A-1. Give an example of a power series $\sum_{k=0}^{\infty} a_k x^k$ that converges for all x with $|x| < 2$ but not if $|x| \ge 2$.

SOLUTION: The geometric series $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{2^k}$

A-2. Let $p(x) = x^3 - 3x + 1$. Show that $p(x)$ has 3 distinct real zeros.

SOLUTION: Observe that $p(-\infty) = -\infty$, $p(0) = 1$, $p(1) = -1$, and $p(+\infty) = +\infty$. Now apply the intermediate value theorem.

One could also exploit that p has critical points at $x = \pm 1$.

A-3. Give an example of a sequence, $f_n(x)$, of bounded functions on the interval $[0, 1]$ that converge pointwise but do *not* converge uniformly. A good sketch is adequate.

SOLUTION: $f_n(x) = x^n$.

A-4. Find a continuous function f and a constant C so that

$$\int_0^x f(t)(1 + t^2)\, dt = x + \cos x + C.$$

SOLUTION: To find C let $x = 0$: $0 = 0 + 1 + C$ so $C = -1$.

To find f use the fundamental theorem of calculus:

$$f(x)(1 + x^2) = 1 - \sin x \quad \text{so} \quad f(x) = \frac{1 - \sin x}{1 + x^2}.$$

A-5. Show that the series. $\displaystyle\sum_{k=0}^{\infty} \frac{1 + \cos 2^k x}{1 + k^4}$ converges uniformly.

SOLUTION: Since $\displaystyle\left|\frac{1 + \cos 2^k x}{1 + k^4}\right| \le \frac{2}{1 + k^4}$, this is a consequence of the Weierstrass M Test.

A-6. Say a function $f(x)$ has the properties $f'(x) = \frac{2x}{1+x^2}$ for all $x \in \mathbb{R}$ and $f(0) = -1$. Show that $f(x) = \ln(1+x^2) - 1$.

SOLUTION: Let $g(x) = f(x) - [\ln(1+x^2) - 1]$. Then $g'(x) = 0$ so $g(x) =$constant. But $g(0) = 0$.

PART B: Two shorter problems, 8 points each [16 points]

B-1. Show that $f(x) = 1/x$ is uniformly continuous in the set $\{x \ge 1\}$.

SOLUTION: Version 1. For all $x, y \ge 1$ we have

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{x-y}{xy}\right| \le |x-y|$$

so given $\epsilon > 0$ pick $\delta = \epsilon$.

Version 2. Because x, $y \ge 1$, then $|f'(x)| = 1/x^2 \le 1$. Then by the Mean Value Theorem

$$|f(x) - f(y)| \le |x - y|$$

so we can let $\delta = \epsilon$.

B-2. Let a_n and b_n be sequences with the properties $a_n \to L$ and $b_n - a_n \to 0$. Given any $\epsilon > 0$, show that $b_n \to L$ by finding an N so that if $n > N$ then $|b_n - L| < \epsilon$.

SOLUTION: Given $\epsilon > 0$.

There is an N_1 so that if $n > N_1$ then $|a_n - L| < \epsilon/2$.

There is an N_2 so that if $n > N_2$ then $|b_n - a_n| < \epsilon/2$.

Let $N = \max\{N_1, N_2\}$. Then for $n > N$

$$|b_n - L| = |b_n - a_n + a_n - L| \le |b_n - a_n| + |a_n - L| \le \epsilon/2 + \epsilon/2 = \epsilon$$

PART C: Four traditional problems, 12 points each [48 points]

C-1. Let $f(x)$ be a continuous function on the interval $I = \{a \le x \le b\}$. and let $\mathcal{P}$ be a partition of I into two intervals having equal width $h = (b-a)/2$. If f is an *increasing* function, Show that the upper and lower Riemann sums satisfy

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) = [f(b) - f(a)]h.$$

[Your solution should include a sketch.]

SOLUTION:

$$\begin{aligned} U(f,\mathcal{P}) =& + f(a+h)h + f(b)h \\ L(f,\mathcal{P}) =& f(a)h + f(a+h)h \end{aligned}$$

Thus

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) = [f(b)) - f(a)]h$$

C-2. a) Let $f(x)$ have two continuous derivatives on $\mathbb{R}$ and let $x_0 < x_1 < x_2$ be given points. If $f(x_0) = f(x_1) = f(x_2) = 0$, show that there is a point $c \in (x_0,\, x_2)$ where $f''(c) = 0$.

SOLUTION: By Rolle's Theorem there is a point $c_1 \in (x_0, x_1$ so that $f'(c_1) = 0$.
Similarly, there is a point $c_2 \in (x_1, x_2$ so that $f'(c_2) = 0$.
Thus there is a point $c \in (c_1, c_2$ so that $f''(c) = 0$.

b) Let $h(x)$ have two continuous derivatives on $\mathbb{R}$ and let $p(x) = Ax^2 + Bx + C$. If

$$h(x_0) = p(x_0), \quad h(x_1) = p(x_1), \quad \text{and} \quad h(x_2) = p(x_2),$$

show there is a point $c \in (x_0,\, x_2)$ where $h''(c) = p''(c) = 2A$.

SOLUTION: Apply part a) to $f(x) := h(x) - p(x)$.

C-3. If f is a continuous function on the interval $[a,\, b]$, let $m := \min_{x\in[a,b]} f(x)$ and $M := \max_{x\in[a,b]} f(x)$.

a) Show that

$$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M.$$

SOLUTION: This is obvious from the Riemann sum definition of the integral.

b) Show there is a point $c \in [a,\, b]$ such that

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$$

REMARK: A useful routine generalization is: for any continuous $w(x) \ge 0$

$$\int_a^b f(x)w(x)\,dx = f(c)\int_a^b w(x),dx$$

SOLUTION Let $Q := \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$. By Part a), $m \le Q \le M$. Thus by the Intermediate Value Theorem there is some $c \in [a, b]$ so that $f(c) = Q$.

C-4. Let $f(x)$ be continuous on the interval $[0,\, 1]$. Show that

$$\lim_{n\to\infty} n\int_0^1 f(x)x^n\,dx = f(1).$$

SOLUTION: [This problem is more difficult than I intended.] Write $J_n(f) = n\displaystyle\int_0^1 f(x)x^n\,dx$.

METHOD 1 To start, note that by a short computation the assertion is true for the special case where $f(x) =$ constant. Write $f(x) = [f(x) - f(1)] + f(1)$, so $J_n(f) = J_n(f(x) - f(1)) + J_n(f(1))$. Since the assertion is true for the constant function $f(1)$, we need only prove it for the function $g(x) := f(x) - f(1)$ which has the additional property that $g(1) = 0$.

Examining the integrand more closely, note that if $0 \le x \le c < 1$ then $nx^n \le nc^n \to 0$ as $n \to \infty$. Thus, all of the action takes place near $x = 1$. This leads us to write

$$J_n(g) = \int_0^1 g(x) nx^n \, dx \;=\; \int_0^c + \int_c^1 \;=\; I_1 + I_2$$

To show that $J_n \to 0$, we will first choose c near 1 so that $|I_2| < \epsilon/2$ for *all* n. We will then pick N so that if $n > N$ then $|I_1| < \epsilon/2$. Assuming for the moment that we have done this, then

$$|J_n| \le |I_1| + |I_2| < \epsilon,$$

as desired.

To estimate I_2, since $g(1) = 0$ we can pick δ so that if $|x - 1| < \delta$ then $|g(x)| < \epsilon/2$ and let $c = 1 - \delta$. With this choice

$$|I_2| \le \int_c^1 |g(x)| \, nx^n \, dx \le (\epsilon/2) \int_c^1 nx^n \, dx < \epsilon/2.$$

To estimate I_1, let $M = \max_{[0,\,1]} |f(x)|$. Then because $0 \le c < 1$, for n large we have

$$|I_1| \le M \int_0^c nx^n \, dx = \frac{n}{n+1} M c^{n+1} < \epsilon/2.$$

Method 2. For the moment we will assume that $f(x)$ is smooth ($f \in C^1([0,1])$ is enough) and integrate by parts:

$$\begin{aligned} J_n(f) n \int_0^1 f(x) x^n \, dx &= \frac{n}{n+1} f(x) x^{n+1} \Big|_0^1 - \frac{n}{n+1} \int_0^1 f'(x) x^{n+1} \, dx \\ &= \frac{n}{n+1} f(1) - \frac{n}{n+1} \int_0^1 f'(x) x^{n+1} \, dx \end{aligned} . \tag{1}$$

To estimate the second term, say $K := \max_{[a,b]} |f'(x)|$. Then

$$\left| \frac{n}{n+1} \int_0^1 f'(x) x^{n+1} \, dx \right| \le K \frac{n}{(n+1)(n+2)}$$

Now let $n \to \infty$ in equation (**??**).

One can apply this even if f is only continuous. Use the fact that there is a smooth function g (even a polynomial)[1] that approximates f uniformly on $[0,\, 1]$:

$$\max_{[a,b]} |f(x) - g(x)| < \epsilon/3$$

The rest is routine:

$$J_n(f) - f(1) = J_n(f - g) + [J_n(g) - g(1)] + [g(1) - f(1)].$$

But

$$|J_n(f - g)| \le J_n(|f - g|) < (\epsilon/3) J_n(1) < \epsilon/3,$$

while, since g is differentiable we know that for n large, $|J_n(g) - g(1)| < \epsilon/3$. Also, $|f(1) - g(1)| < \epsilon/3$.

[1] Weierstrass Approximation Theorem