Math 202 | **Exam 3** | Jerry L. Kazdan
December 6, 2018 | 10:30–11:500

DIRECTIONS: Part A has 6 short questions (5 points each), Part B has 2 shorter problems (8 points each), Part C has 4 traditional problems (12 points each). 94 points total].
To receive full credit your solution should be clear and correct. Neatness counts. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3 ×5 with notes on both sides.

PART A: Six shorter problems, 5 points each [total: 30 points]

A-1. Give an example of a power series $\sum_{k=0}^{\infty} a_k x^k$ that converges for all x with $|x| < 2$ but not if $|x| \geq 2$.

A-2. Let $p(x) = x^3 - 3x + 1$. Show that $p(x)$ has 3 distinct real zeros.

A-3. Give an example of a sequence, $f_n(x)$, of bounded functions on the interval [0, 1] that converge pointwise but do *not* converge uniformly. A good sketch is adequate.

A-4. Find a continuous function f and a constant C so that

$$\int_0^x f(t)(1+t^2)\, dt = x + \cos x + C.$$

A-5. Show that the series $\displaystyle\sum_{k=0}^{\infty} \frac{1 + \cos 2^k x}{1 + k^4}$ converges uniformly.

A-6. Say a function $f(x)$ has the properties $f'(x) = \frac{2x}{1+x^2}$ for all $x \in \mathbb{R}$ and $f(0) = -1$. Show that $f(x) = \ln(1 + x^2) - 1$.

PART B: Two shorter problems, 8 points each [16 points]

B-1. Show that $f(x) = 1/x$ is uniformly continuous in the set $\{x \geq 1\}$.

B-2. Let a_n and b_n be sequences with the properties $a_n \to L$ and $b_n - a_n \to 0$. Given any $\epsilon > 0$, show that $b_n \to L$ by finding an N so that if $n > N$ then $|b_n - L| < \epsilon$.

PART C: Four traditional problems, 12 points each [48 points]

C-1. Let $f(x)$ be a continuous function on the interval $I = \{a \leq x \leq b\}$. and let $\mathcal{P}$ be a partition of I into two intervals having equal width $h = (b - a)/2$. If f is an *increasing* function, Show that the upper and lower Riemann sums satisfy

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) = [f(b) - f(a)]h.$$

[Your solution should include a sketch.]

C-2. a) Let $f(x)$ have two continuous derivatives on $\mathbb{R}$ and let $x_0 < x_1 < x_2$ be given points. If $f(x_0) = f(x_1) = f(x_2) = 0$, show that there is a point $c \in (x_0, x_2)$ where $f''(c) = 0$.

b) Let $h(x)$ have two continuous derivatives on $\mathbb{R}$ and let $p(x) = Ax^2 + Bx + C$. If

$$h(x_0) = p(x_0), \quad h(x_1) = p(x_1), \quad \text{and} \quad h(x_2) = p(x_2),$$

show there is a point $c \in (x_0, x_2)$ where $h''(c) = p''(c) = 2A$.

C-3. If f is a continuous function on the interval $[a, b]$, let $m := \min_{x\in[a,b]} f(x)$ and $M := \max_{x\in[a,b]} f(x)$.

a) Show that

$$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M.$$

b) Show there is a point $c \in [a, b]$ such that

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$$

C-4. Let $f(x)$ be continuous on the interval $[0, 1]$. Show that

$$\lim_{n\to\infty} n\int_0^1 f(x)x^n\,dx = f(1).$$