Signature | Printed Name

Math 202
November 1, 2018

Exam 2

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10:30 — 11:50

Directions: Part A has 8 shorter problems (5 points each) while Part B has 4 traditional problems (10 points each). To receive full credit your solution should be clear and correct. Neatness counts. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3 ×5 with notes on both sides.

Part A: Eight shorter problems, 5 points each.

A-1. Find all points in the complex plane where $\sum_0^\infty \frac{n}{(z-2)^n}$ converges.

Score	
A-1	
A-2	
A-3	
A-4	
A-5	
A-6	
A-7	
A-8	
B-1	
B-2	
B-3	
B-4	
Total	

A-2. This problem concerns the continuity of $f(x) = \frac{1}{x}$ at the point $a = 1/1000$, Let $\epsilon = 1$. Find a $\delta > 0$ so that if $|x-a| < \delta$ then $|f(x) - f(a)| < \epsilon$.

A-3. Give an example of a bounded continuous function $f(x)$, $x \in \mathbb{R}$, that does *not* attain its infimum. A clear sketch is adequate.

A-4. Say a function $f(x)$ has the properties $f'(x) = 2\cos 2x$ for all $x \in \mathbb{R}$ and $f(0) = 0$. Show that $f(x) = \sin 2x$.

[HINT: To show that "$A = B$", it is often easiest to let $C = A - B$ and then show that "$C = 0$".]

A-5. Let $f(x)$ and $g(x)$ be continuous on $[a, b]$. If $f(a) > g(a)$ and $f(b) < g(b)$, prove that there is some $c \in (a, b)$ where $f(c) = g(c)$.

Can there be more than one such point?

A-6. Give an example of a function $f(x)$ that is continuous at every point of the set $\{x \geq 1\}$ but is not uniformly continuous in this set.

A-7. Give an example of a function $f(x)$ that is continuous for $-1 \le x \le 1$ but not differentiable at, say, $x = 0$.

A-8. Let $f(x)$, $g(x)$, and $h(x)$ be smooth functions

a) If $f(a) = 0$ and $f'(x) \ge 0$ for all $x \ge a$, show that $f(x) \ge f(a)$ for all $x \ge a$.

b) If $g(a) = h(a)$ and $g'(x) \ge h'(x)$ for all $x \ge a$, show that $g(x) \ge h(x)$ for all $x \ge a$.

PART B: Four traditional problems, 10 points each.

B-1. Use the definition of the derivative as the limit of a difference quotient to show that if $f(x) = \cos 2x$, then f is differentiable everywhere and compute its derivative. [You may use that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ and $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = 0$.]

B-2. Let $f(x)$ be a smooth function with the properties $f(0) = 3$, $f)1) = 1$, and $f(3) = 5$. Show that $f''(c) \geq A > 0$ for some $c \in (0, 3)$ and some $A > 0$. Give an explicit value for the constant A.

B-3. Let $f(x)$ be differentiable at every point of the open interval $a < x < b$ (possibly unbounded).

a) If the derivative is bounded, say $|f'(x)| \le M$, in this interval, show that f is uniformly continuous in the interval.

b) If the derivative is **not** bounded in this interval, show that f is **not** uniformly continuous in the interval.

COUNTEREXAMPLES All of these are uniformly continuous:

$f(x) = \sqrt{x}$ for $0 \le x \le 1$.

$g(x) = x\sin(1/x)$ for $0 < x \le 1$, $g(0) = 0$,

$h(x) = \frac{\sin x^3}{x}$ for $1 \le x$.

c) Apply these to the functions x^2 and $1/x$ on the interval $x \ge 1$.

B-4. a) Say the smooth function $w(x)$ satisfies $w'' - c(x)w \le 0$, where $c(x) > 0$. Show there is no point p where w has a local minimum and $w(p) < 0$.

b) If on a bounded interval $a \le x \le b$ w satisfies this and $w(a) = w(b) = 0$, show that $w(x) \ge 0$ on the whole interval.

c) Say on the interval $[a, b]$ the smooth functions $u(x)$ and $v(x)$ satisfy

$$u'' - c(x)u = f(x), \qquad v'' - c(x)v = g(x), \qquad \text{with} \quad u(a) = v(a), \quad u(b) = v(b),$$

where, as above, $c(x) > 0$, and f and g are given functions. If $f(x) \le g(x)$, show that $u(x) \ge v(x)$ in $[a, b]$.

[Extra page for scratch]