Math 202 **Exam 2** Jerry L. Kazdan
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DIRECTIONS: Part A has 8 shorter problems (5 points each) while Part B has 4 traditional problems (10 points each). To receive full credit your solution should be clear and correct. Neatness counts. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3 ×5 with notes on both sides.

PART A: Eight shorter problems, 5 points each.

A-1. Find all points in the complex plane where $\sum_0^\infty \frac{n}{(z-2)^n}$ converges.

SOLUTION: By the ratio test, this converges absolutely when $\left|\frac{1}{z-2}\right| < 1$, that is, when $|z-2| > 1$. This is the *exterior* of a disk centered at $z=2$ with radius 1.

The series diverges at every point of the boundary of this disk since at these points $n/|z-2|^n = n$. This uses: "If a series $\sum c_n$ converges then $|c_n| \to 0$."

A-2. This problem concerns the continuity of $f(x) = \frac{1}{x}$ at the point $a = 1/1000$. Let $\epsilon = 1$.

Find a $\delta > 0$ so that if $|x-a| < \delta$ then $|f(x) - f(a)| < \epsilon$.

SOLUTION: We want a δ so that if $|x - \frac{1}{1000}| < \delta$, then $|\frac{1}{x} - 1000| < 1$, that is, $999 < \frac{1}{x} < 1001$; equivalently, $\frac{1}{1001} < x < \frac{1}{999}$. This means

$$\frac{1}{1001} - \frac{1}{1000} < x - a < \frac{1}{999} - \frac{1}{1000},$$

which is satisfied if $\delta < \frac{1}{1000} - \frac{1}{1001} = \frac{1}{1{,}001{,}000}$. To be less exact, we can let $\delta = 10^{-7}$. .

A-3. Give an example of a bounded continuous function $f(x)$, $x \in \mathbb{R}$, that does *not* attain its infimum. A clear sketch is adequate.

SOLUTION: $\frac{1}{1+x^2}$.

A-4. Say a function $f(x)$ has the properties $f'(x) = 2\cos 2x$ for all $x \in \mathbb{R}$ and $f(0) = 0$. Show that $f(x) = \sin 2x$. [HINT: To show that "$A = B$", it is often easiest to let $C = A - B$ and then show that "$C = 0$".]

SOLUTION: Let $h(x) := f(x) - \sin 2x$.

I show that $h(x) = 0$. First, $h'(x) = 0$ so $h(x) = \text{constant}$. But $h(0) = 0$.

A-5. Let $f(x)$ and $g(x)$ be continuous on $[a,b]$. If $f(a) > g(a)$ and $f(b) < g(b)$, prove that there is some $c \in (a,b)$ where $f(c) = g(c)$.

Solution Let $h(x) := f(x) - g(x)$ and note that $h(a) > 0$ while $h(b) < 0$. Now apply the intermediate value theorem.

Can there be more than one such point?

Solution Yes, lots. Look at the graphs of $f(x) = \cos x$ and $g(x) = \sin x$ for $0 \le x \le 3\pi$

A-6. Give an example of a function $f(x)$ that is continuous at every point of the set $\{x \ge 1\}$ but is not uniformly continuous in this set.

Solution x^2

A-7. Give an example of a function $f(x)$ that is continuous for $-1 \le x \le 1$ but not differentiable at, say, $x = 0$.

Solution $|x|$

A-8. Let $f(x)$, $g(x)$, and $h(x)$ be smooth functions

a) If $f(a) = 0$ and $f'(x) \ge 0$ for all $x \ge a$, show that $f(x) \ge f(a)$ for all $x \ge a$.

Solution By the Mean Value Theorem there is a point c, $a < c < x$ so that

$$f(x) - f(a) = f'(c)(x - a)$$

Since $f'(c) \ge 0$, then $f(x) - f(a) \ge 0$.

b) If $g(a) = h(a)$ and $g'(x) \ge h'(x)$ for all $x \ge a$, show that $g(x) \ge h(x)$ for all $x \ge a$.

Solution Let $f(x) = g(x) - h(x)$ and apply part a).

Part B: Four traditional problems, 10 points each.

B-1. Use the definition of the derivative as the limit of a difference quotient to show that if $f(x) = \cos 2x$, then f is differentiable everywhere and compute its derivative. [You may use that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ and $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0$.]

B-2. Let $f(x)$ be a smooth function with the properties $f(0) = 3$, $f)1) = 1$, and $f(3) = 5$. Show that $f''(c) \ge A > 0$ for some $c \in (0, 3)$ and some $A > 0$. Give an explicit value for the constant A.

Solution: By the Mean value Theorem for the intervals $0 \le x \le 1$ and $1 \le x \le 3$ there are points $c_1 \in (0, 1)$ and $c_2 \in (1, 3)$ so that

$$f'(c_1) = \frac{f(1) - f(0)}{1 - 0} = -2 \qquad f'(c_2) = \frac{f(3) - f(1)}{3 - 1} = 2.$$

Now apply the Mean Value Theorem again to $f'(x)$ for $c_1 < x < c_2$ to find a $c_3 \in (c_1, c_2)$ do that

$$f''(c_3) = \frac{f'(c_2) - f'(c_1)}{c_2 - c_1} = \frac{4}{c_2 - c_1} \ge \frac{4}{3}.$$

B-3. Let $f(x)$ be differentiable at every point of the open interval $a < x < b$ (possibly unbounded).

a) If the derivative is bounded, say $|f'(x)| \le M$, in this interval, show that f is uniformly continuous in the interval.

b) If the derivative is **not** bounded in this interval, show that f is **not** uniformly continuous in the interval.

SOLUTION: This assertion is *FALSE*. All of the counterexamples below are uniformly continuous – although their first derivatives are unbounded:

$f(x) = \sqrt{x}$ for $0 \le x \le 1$ (the simplest example).

$g(x) = x \sin(1/x)$ for $0 < x \le 1$, $g(0) = 0$,

$h(x) = \frac{\sin x^3}{x}$ for $1 \le x$.

c) Apply these to the functions x^2 and $1/x$ on the interval $x \ge 1$.

B-4. a) Say the smooth function $w(x)$ satisfies $w'' - c(x)w \le 0$, where $c(x) > 0$. Show there is no point p where w has a local minimum and $w(p) < 0$.[1]

SOLUTION: At a local minimum $w'' \ge 0$. Since $w(p) < 0$ and $c(x) > 0$, this contradicts $w'' - c(x)w \le 0$.

b) If on a bounded interval $a \le x \le b$ w satisfies this and $w(a) = w(b) = 0$, show that $w(x) \ge 0$ on the whole interval.

SOLUTION: Reasoning by contradiction, say $w(p) < 0$ somewhere in $[a, b]$. Let q be the point in $[a, b]$ where w has its *minimum* value. Then $w(q) < 0$. Note, q can't be an endpoint because $w(a) = w(b) = 0$. Therefore w has a negative local minimum at q. By part a), this is impossible. Therefore $w(x) \ge 0$ for all $x \in [a, b]$.

c) Say on the interval $[a, b]$ the smooth functions $u(x)$ and $v(x)$ satisfy

$$u'' - c(x)u = f(x), \qquad v'' - c(x)v = g(x), \qquad \text{with} \quad u(a) = v(a), \quad u(b) = v(b),$$

where, as above, $c(x) > 0$, and f and g are given functions. If $f(x) \le g(x)$, show that $u(x) \ge v(x)$ in $[a, b]$.

SOLUTION: Let $w(x) := u(x) - v(x)$. Then by part b) we have $w(x) \ge 0$.

[1] This whole problem – with the *same* proof – is valid for the more general differential operator $w'' + b(x)w' - c(x)w$, where $b(x)$ is any continuous function.