Math 202 | Exam 1 | Jerry L. Kazdan
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Exam 1

DIRECTIONS: Part A has 8 shorter problems (5 points each) while Part B has 4 traditional problems (10 points each). To receive full credit your solution should be clear and correct. Neatness counts. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3 ×5 with notes on both sides.

PART A: Eight shorter problems, 5 points each.

A-1. If $a_1 = 1$ and $a_{n+1} = \sqrt{3a_n + 4}$, show that $a_n < 4$ for all $n = 1, 2, 3, \ldots$.

SOLUTION: Use induction.

A-2. Show that $\sqrt{3}$ is not a rational number.

SOLUTION: Routine

A-3. Show that $\lim_{n\to\infty} \dfrac{5^n}{n!} = 0$

SOLUTION: Observe that $a_{11} = \frac{5^{11}}{11!} = \frac{5^{10}}{10!}\frac{5}{11} < a_{10}/2$

Similarly, $a_{12} < a_{11}/2 < a_{10}/2^2$ Each of the subsequent terms is less than $1/2$ its predicessor, that is, for $n \ge 10$: $0 < a_{n+1} < a_n/2$. Thus the sequence converges to 0.

OR: use the ratio test for sequences: $|a_{n+1}/a_n| = 5/(n+1) \to 0 < 1$.

A-4. Give an example of a sequence of real numbers that is not monotone but that converges to some limit.

SOLUTION: Two examples: $a_n = (-1)^n/n \to 0$, $\quad b_n = 5^n/n! \to 0$.

A-5. Give an example of a sequence x_n of real numbers with at least two subsequences that converge to different limits.

SOLUTION: Two examples: $a_n = (-1)^n$, $\quad b_n = (-1)^n(2 + \frac{1}{n})$

A-6. Give an example of an *unbounded* sequence of real numbers a_n that satisfies $|a_{n+1} - a_n| \to 0$.

SOLUTION: Three examples: $a_n = \sqrt{n}$, $\quad b_n = \ln n$, $\quad c_n = 1 + 1/2 + 1/3 + \cdots + 1/n$.

A-7. Determine if the series $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1} + \cdots$ converges or diverges. Explain your reasoning.

SOLUTION:

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1} > \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n+2}$$
$$= \frac{1}{2}\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n+1}\right] \to \infty$$

Equivalently:

$$\sum_0^n \frac{1}{2k+1} > \sum_0^n \frac{1}{2k+2} = \frac{1}{2}\sum_0^n \frac{1}{k+1}$$

which diverges to infinity.

A-8. [PROOF OR COUNTEREXAMPLE] Let a_n be a sequence of real numbers that converges to L. Then there exists an $\epsilon > 0$ such that for all integers n we have $|a_{n+1} - a_n| < \epsilon$.

SOLUTION: The key is that the inequality

$|a_{n+1} - a_n| < \epsilon$ needs to hold for *all* n, not just for n sufficiently large. Since convergence concerns only terms with n sufficiently large, we don't have much control over the terms at the "beginning" of the sequence. But since the sequence converges, we at least know that it is bounded: for some M we have $|a_n| < M$ for *all* n.

At this stage, for me it is simplest to rename ϵ to, say, Q. Thus:

"Then there exists a $Q > 0$ such that for all integers n we have $|a_{n+1} - a_n| < Q$."

This assertion is correct. We see that all that is required is that the sequence be bounded, say $|a_n| < M$. Then $|a_{n+1} - a_n| < |a_{n+1}| + |a_n| \le 2M$ so we can pick $Q = 2M$.

PART B: Four traditional problems, 10 points each.

B-1. Let a_n and b_n be sequences of of complex numbers. If $a_n \to A$ and $b_n \to B$, show that $a_n b_n \to AB$. [Give a formal proof using ϵ and N.]

SOLUTION: Standard (and essential) preliminary:

$$|a_n b_n - AB| = |(a_n - A)b_n + A(b_n - B)| \le |a_n - A||b_n| + |A||b_n - B|.$$

Now one needs the key ingredient that since the sequence b_n converges, it is bounded: $|b_n| < M$ for some M.

The rest is routine: Given $\epsilon > 0$, pick N so that if $n > N$ then $|a_n - A|M < \epsilon/2$ and $|A|||b_n - B| < \epsilon/2$. Then, by the above:

$$|a_n b_n - AB| < \epsilon/2 + \epsilon/2 = \epsilon.$$

B-2. Let $a_n = \sqrt{n^2 + 6n} - n$. Show that a_n converges and find the limit.

SOLUTION: A standard device:

$$a_n = (\sqrt{n^2+6n} - n)\left(\frac{\sqrt{n^2+6n}+n}{\sqrt{n^2+6n}+n}\right) = \frac{(n^2+6n) - n^2}{\sqrt{n^2+6n}+n} = \frac{6n}{\sqrt{n^2+6n}+n} \to 3.$$

B-3. Let a_n be a sequence of real numbers that converge to L. If $L > 0$, show there is an N so that if $n > N$ then $a_n > \frac{1}{2}L$.

SOLUTION: Pick N so that if $n > N$ then $|a_n - L| < \frac{1}{2}L$. In particular, $-\frac{1}{2}L < a_n - L$, that is, $\frac{1}{2}L < a_n$.

B-4. Let $a_n \to A$ and $b_n \to B$ be convergent sequences of real numbers, and let c_n be the larger of a_n and b_n, so $c_n = \max(a_n, b_n)$. Either prove that this sequence c_n converges or give a counterexample.

REMARK: There are three cases: $A < B$, $A > B$, and $A = B$.

SOLUTION: If $A < B$, then for all sufficiently large n we have $a_n < b_n$ so $c_n = b_n \to B$.

The case $A > B$ is essentially identical.

If $A = B$, given $\epsilon > 0$ pick N so that if $n > N$ then both $|a_n - A| < \epsilon$ and $|b_n - A| < \epsilon$. Since either $c_n = a_n$ or $c_n = b_n$ then $|c_n - A| < \epsilon$.