$$\boxed{\sin x + \sin 2x + \cdots + \sin nx = \frac{\cos\frac{x}{2} - \cos(n+\frac{1}{2})x}{2\sin\frac{x}{2}}}$$

The key to obtaining this formula is either to use some imaginative trigonometric identities or else recall that $e^{ix} = \cos x + i\sin x$ and then routinely sum a geometric series. I prefer the later. Thus

$$\sin x + \sin 2x + \cdots + \sin nx = \operatorname{Im}\{e^{ix} + e^{i2x} + \cdots + e^{inx}\}, \tag{1}$$

where $\operatorname{Im}\{z\}$ means take the imaginary part of the complex number $z = x + iy$. The sum on the right side is a (finite) geometric series $t + t^2 + \cdots t^n$ where $t = e^{ix}$:

$$t + t^2 + \cdots + t^n = \frac{t(1-t^n)}{1-t}.$$

Thus

$$\sin x + \sin 2x + \cdots + \sin nx = \operatorname{Im}\left\{\frac{e^{ix}(1-e^{inx})}{1-e^{ix}}\right\}. \tag{2}$$

We need to find the imaginary part of the fraction on the right. The denominator is what needs work. By adding and subtracting

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{and} \quad e^{-i\theta} = \cos\theta - i\sin\theta$$

we obtain the important formulas

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Thus

$$1 - e^{ix} = e^{ix/2}\left(e^{-ix/2} - e^{ix/2}\right) = -2ie^{ix/2}\sin\frac{x}{2}$$

so

$$\frac{e^{ix}(1-e^{inx})}{1-e^{ix}} = i\left[\frac{e^{i\frac{x}{2}} - e^{i(n+\frac{1}{2})x}}{2\sin\frac{x}{2}}\right]. \tag{3}$$

Consequently, from (3), taking the imaginary part of the right side (so the real part of $[\cdots]$) we obtain the desired formula:

$$\sin x + \sin 2x + \cdots + \sin nx = \frac{\cos\frac{1}{2}x - \cos(n+\frac{1}{2})x}{2\sin\frac{1}{2}x}$$

Remark: By taking the real part in (3) we obtain the related formula

$$\cos x + \cos 2x + \cdots + \cos nx = \frac{-\sin\frac{1}{2}x + \sin(n+\frac{1}{2})x}{2\sin\frac{1}{2}x}.$$

Exercise: Use $\sin(a+x)+\sin(a+2x)+\cdots+\sin(a+nx) = \mathrm{Im}\{e^{ia}(e^{ix}+\cdots+e^{inx})\}$ to compute a formula for $\sin(a+x)+\sin(a+2x)+\cdots+\sin(a+nx)$. [Taking the derivative of this formula with respect to a gives another route to the formula of the Remark just above.]

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