ODE: Existence and Uniqueness of a Solution

The Fundamental Theorem of Calculus tells us how to solve the ordinary differential equation (ODE)

$$\frac{du}{dt} = f(t) \qquad \text{with initial condition} \qquad u(0) = \alpha.$$

Just integrate both sides:

$$u(t) = \alpha + \int_0^t f(s)\,ds.$$

It is not obvious how to solve

$$\frac{du(t)}{dt} = f(x, u(t)) \qquad \text{with initial condition} \qquad u(0) = \alpha$$

because the unknown, $u(t)$, is on both sides of the equation. In many particular cases, by using special devices one can find formulas for the solutions – but it is far from obvious that a solution exists or is unique. In fact, there are simple examples showing that unless one is careful, a solution may not exist, and even if one exists, it may not be unique. Just because one may want something to happen doesn't mean that this will happen. It is easy to have presumptions that turn out to be not quite true.

The results that we present are classical. They will use most of the ideas we have covered this semester.

We investigate one illuminating case and will prove the existence and uniqueness of a solution of the system of inhomogeneous linear equation

$$\frac{d\vec{U}(t)}{dt} = A(t)\vec{U}(t) + \vec{F}(t) \qquad \text{with} \qquad \vec{U}(0) = \vec{\alpha}. \tag{1}$$

Here we seek a vector $\vec{U}(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t)) \end{pmatrix}$ given the input $n \times n$ matrix $A(t) = (a_{ij}(t))$, and a vector $\vec{F}(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t)) \end{pmatrix}$. The elements of $A(t)$ and $F(t)$ are assumed to depend continuously on t for $|t| \le b$. Also $\alpha \in \mathbb{R}^n$ is the initial condition.

Although we will not pursue it, there is fairly straightforward extension of the method we use to the more general nonlinear case

$$\frac{d\vec{U}}{dt} = \vec{F}(t, \vec{U}(t)) \qquad \text{with} \qquad \vec{U}(0) = \vec{\alpha}.$$

The ideas are already captured in our special case of equation (1).

Example Here we have a system of two equations

$$\begin{aligned} u_1'(t) &= -\,u_2(t) \\ u_2'(t) &= \;\; u_1(t) \end{aligned}$$

with initial conditions $u_1(0) = 1$ and $u_2(0) = 0$. The (unique) solution of this happens to be $u_1(t) = \cos t$, $u_2(t) = \sin t$, but the point of these notes is to consider equations where there may *not* be simple formulas for the solution.

The primary reason we are presenting the more general matrix case $n \geq 1$ is apply to the standard second order scalar initial value problem

$$y''(t) + p(t)y'(t) + q(t)y(t) = f(t) \quad \text{with} \quad y(0) = a \quad \text{and} \quad y'(0) = b, \tag{2}$$

where $p(t)$, $q(t)$, and $f(t)$ are continuous real-valued functions.
To reduce the problem (2) to problem (1), let $u_1 = y$ and $u_2 = y'$. then

$$\begin{aligned} u_1' &= y' = u_2 \\ u_2' &= y'' = -pu_2 - qu_1 + f \end{aligned}$$

that is,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{3}$$

This exactly has the form of the system (1). If we can solve the system (3), then $u_1(t)$is the solution of equation (2)

Example Before plunging ahead to prove that equation (1) has exactly one solution, we present the idea with the simple example

$$u' = u \qquad \text{with} \qquad u(0) = 1 \tag{4}$$

whose solution we already know is $u(t) = e^t$. The first step is to integrate both sides of equation (4) to obtain the equivalent problem of finding a function $u(t)$ that satisfies

$$u(t) = 1 + \int_0^t u(s)\, ds. \tag{5}$$

We will solve this by *successive approximations.* Let the initial approximation be $u_0(t) = u(0) = 1$ and define the subsequent approximations by the rule

$$u_{k+1}(t) = 1 + \int_0^t u_k(s)\, ds, \qquad k = 0, 1, 2, \ldots. \tag{6}$$

Then

$$
\begin{aligned}
u_1(t) &= 1 + \int_0^t 1\, ds = 1 + t \\
u_2(t) &= 1 + \int_0^t (1+s)\, ds = 1 + t + \tfrac{1}{2}t^2 \\
u_3(t) &= 1 + \int_0^t (1 + s + \tfrac{1}{2}s^2)\, ds = 1 + t + \tfrac{1}{2}t^2 + \tfrac{1}{3!}t^3 \\
&\ \vdots \\
u_k(t) &= 1 + t + \tfrac{1}{2}t^2 + \cdots + \tfrac{1}{k!}t^k
\end{aligned}
$$

We clearly recognize the Taylor series for e^t emerging. This gives us hope that our successive approximation approach to equation (6) is a plausible technique.

With this as motivation we integrate equation (1) and obtain

$$\vec{U}(t) = \vec{\alpha} + \int_0^t \left[A(s)\vec{U}(s) + \vec{F}(s) \right] ds \tag{7}$$

We immediately observe that if a continuous function $\vec{U}(t)$ satisfies this, then by the Fundamental Theorem of Calculus applied to the left side, this $\vec{U}(t)$ is differentiable and is a solution of our equation (1). Therefore we need only find a continuous $\vec{U}(t)$ that satisfies equation (7).

As in our example we use successive approximations. To make the issues clearer, I will give the proof *twice*. First for the special case $n = 1$

$$u'(t) = a(t)u(t) + f(t) \qquad \text{with} \qquad u(0) = \alpha \tag{8}$$

so there are no vectors or matrices. In this particularly special case there happens to be an explicit formula for the solution – but we won't use it since it will not help for the general matrix case. We are assuming that $a(t)$ and $f(t)$ are continuous on the interval $[0, b]$. Consequently, they are bounded there and we use the uniform norm

$$\|a\| := \sup_{t\in[0,b]} |a(t)|, \qquad \|f\| := \sup_{t\in[0,b]} |f(t)|, \tag{9}$$

Just as with equation (1) we integrate (8) to find

$$u(t) = \alpha + \int_0^t [a(s)u(s) + f(s)]\, ds. \tag{10}$$

As we observed after equation (7), if we have a continuous function $u(t)$ that satisfies this, then by the Fundamental Theorem of Calculus the $u(t)$ on the left side is differentiable and is the desired solution of equation (8).

For our initial approximation let $u_0(t) = 0$ (this particular choice is not very important) and recursively define

$$u_{k+1}(t) = \alpha + \int_0^t [a(s)u_k(s) + f(s)]\, ds, \qquad k = 0, 1, \ldots. \tag{11}$$

We will show that the u_k converge uniformly to the desired solution of equation (10). While the optimal version is to do this on the whole interval $[0, b]$, it is simpler (and, in many ways more illuminating) to work assuming t is in the smaller interval $[0, \beta]$, where $\beta < \min(b, 1/\|a\|)$. Note that since $u_k(t)$ is continuous in the interval $[0, \beta]$, so is $u_{k+1}(t)$. The uniform norms $\|u\|$, we use for the u_j will now be as in equation, except on this smaller interval $[0, \beta]$. Subtracting we obtain

$$\begin{aligned} u_{k+1}(t) - u_k(t) &= \int_0^t a(s)[u_k(s) - u_{k-1}(s)]\, ds \\ &\le \|a\| \|u_k - u_{k-1}\| \beta \le c\|u_k - u_{k-1}\|, \end{aligned} \tag{12}$$

where $c := \|a\|\beta < 1$. Because the right side is independent of t, the sequence of successive approximations is *contracting*

$$\|u_{k+1} - u_k\| \le c\|u_k - u_{k-1}\|. \tag{13}$$

Using this inequality repeatedly we deduce that

$$\|u_{k+1} - u_k\| \le c\|u_k - u_{k-1}\| \le \cdots \le c^k \|u_1 - u_0\|. \tag{14}$$

Since $0 < c < 1$, the series $\sum c^k \|u_1 - u_0\|$ converges. Thus by the Weierstrass M-test the series $\sum |u_{k+1}(t) - u_k(t)|$ of continuous functions converges absolutely and uniformly in the interval $[0, \beta]$ to some continuous function. But

$$\begin{aligned} \sum_{k=0}^N [u_{k+1}(t) - u_k(t)] &= [u_{N+1}(t) - u_N(t)] + [u_N(t) - u_{N-1}(t)] + \cdots + [u_1(t) - u_0(t)] \\ &= u_{N+1}(t) - u_0(t) = u_{N+1}(t) \end{aligned}$$

Consequently the sequence of continuous functions $u_N(t)$ converges uniformly in the interval $[0, \beta]$ to some continuous function $u(t)$. We use this to let $k \to \infty$ in equation (11) and find that $u(t)$ is the desired solution of equation (10). In this last step we used the uniform convergence to interchange limit and integral in equation (11).

This completes the existence proof for the special case of equation (8).

We now repeat the proof for the general system equation (7), using, where needed, standard facts about vectors and matrices from the Appendix at the end of these notes. It is remarkable that the only changes needed are changes in notation.

Just as in equation (11) we let $\vec{U}_0(t) = \vec{0}$ and recursively define

$$\vec{U}_{k+1}(t) = \vec{\alpha} + \int_0^t \left[A(s)\vec{U}_k(s) + \vec{F}(s)\right] ds, \qquad k = 0, 1, \ldots.$$

Given the continuous $\vec{U}_k(t)$ this defines the next approximation, $\vec{U}_{k+1}(t)$. We will show that the $\vec{U}_k$ converge uniformly to the desired solution of equation (7) in the smaller interval $[0,\, \beta]$, where $\beta = \min(b,\, 1/\|A\|$. Now

$$\vec{U}_{k+1}(t) - \vec{U}_k(t) = \int_0^t A(s)\left[\vec{U}_k(s) - \vec{U}_{k-1}(s)\right] ds$$

To estimate the right hand side we use the inequalities (22) and (21) from the Appendix

$$\begin{aligned} |\vec{U}_{k+1}(t) - \vec{U}_k(t)| \le & \|A\| \int_0^t \left|\vec{U}_k(s) - \vec{U}_{k-1}(s)\right| ds \\ \le & \|A\| \|\vec{U}_k - \vec{U}_{k-1}\| \beta = c\|\vec{U}_k - \vec{U}_{k-1}\| \end{aligned}, \tag{15}$$

where $c = \|A\|\beta < 1$.

The remainder of the proof goes exactly as in the previous special case and proves the existence of a solution to equation (7) and hence our differential equation (1).

Uniqueness There are several ways to prove the uniqueness of the solution of the initial value problem (1). None of them are difficult. We work in the interval $[0,\, \beta]$ defined above. Say $\vec{U}(t)$ and $\vec{V}(t)$ are both solutions. Let $\vec{W}(t) := \vec{U}(t) - \vec{V}(t)$. Then $\vec{W}' = A\vec{W}$ with $\vec{W}(0) = 0$. We want to show that $W(t) \equiv 0$.
Just as in equation (7)

$$\vec{W}(t) = \int_0^t A(t)\vec{W}(t)\, dt.$$

Thus, similar to the computation in (15)

$$|\vec{W}(t)| \le \|A\| \|\vec{W}\| \beta \;=\; c\|\vec{W}\|.$$

Because the right-hand side is independent of $t \in [0.\, \beta]$,

$$\|\vec{W}\| \le c\|\vec{W}\|.$$

Because $0 < c < 1$, this implies that $\|\vec{W}\| = 0$, Thus $\vec{U}(t) = \vec{V}(t)$ for $t \in [0.\, \beta]$.

Remark: There is a useful conceptual way to think of the proof. If $v(t)$ is a continuous function, define the map $T : C([0,\, b) \to C([0,\, b])$ by the rule

$$T(u)(t) := \alpha + \int_0^t [a(s)u(s) + f(s)]\, ds. \tag{16}$$

Then equation (10) says the the solution we are seeking satisfies $u = T(u)$. In other words u is a *fixed point* of the map T. The solution to many questions can be usefully attacked by viewing them as seeking a fixed point of some map.
For this particular map, T, it is illuminating to note that for any continuous functions u and V on the interval $[0, \beta]$

$$T(u)(t) - T(v)(t) = \int_0^t a(s)[u(s) - v(s)]\, ds$$

so that

$$|T(u)(t) - T(v)(t)| \le \|a\|\|u - v\|\beta.$$

Because the right side in independent of t and $\|a\|\beta = c < 1$, we conclude that

$$\|T(u) - T(v)\| \le c\|u - v\|.$$

Because $c < 1$ we see that T contracts distance between function. In this situation, T is called a *contraction mapping.* These arise frequently.

Appendix on Norms of Vectors and Matrices

This is a review of a few items concerning vectors and matrices. Let $\vec{u} := (u_1, u_2, \ldots, u_n)$ and $\vec{v} := (v_1, \ldots, v_n)$ be points (vectors) in $\mathbb{R}^n$. Their *inner product* (also called their *dot product* is defined as

$$\langle \vec{u},\, \vec{v} \rangle = \langle \vec{u},\, \vec{v} \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

In particular, the Euclidean length

$$|\vec{u}|^2 = \langle \vec{u},\, \vec{u} \rangle = u_1^2 + u_2^2 + \cdots + u_n^2.$$

This gives the useful formula

$$\begin{aligned} |\vec{u} - \vec{v}|^2 =& \langle \vec{u} - \vec{v},\, \vec{u} - \vec{v} \rangle = \langle \vec{u},\, \vec{u} \rangle - 2\langle \vec{u},\, \vec{v} \rangle + \langle \vec{v},\, \vec{v} \rangle \\ =& |\vec{u}|^2 - 2\langle \vec{u},\, \vec{v} \rangle + |\vec{v}|^2 \end{aligned} \tag{17}$$

For vectors in the plane, $\mathbb{R}^2$, the inner product is interpreted geometrically as

$$\langle \vec{u},\, \vec{v} \rangle = |\vec{u}||\vec{v}| \cos\theta,$$

where θ is the angle between $\vec{u}$ and $\vec{v}$. Since $|\cos\theta| \le 1$, this implies the *Cauchy inequality*

$$|\langle \vec{u},\, \vec{v} \rangle| \le |\vec{u}||\vec{v}|. \tag{18}$$

The following is a direct analytic proof the Cauchy inequality in $\mathbb{R}^n$ without geometric considerations – which gave us valuable insight. We begin by noting that using the inner product, from equation (17) for any real number t

$$0 \le |\vec{u} - t\vec{v}|^2 = |\vec{u}|^2 - 2t\langle \vec{u},\, \vec{v} \rangle + t^2|\vec{v}|^2. \tag{19}$$

Pick t to that the right side is as small as possible (so take the derivative with respect to t). We find $t = \langle \vec{u}, \vec{v} \rangle / |\vec{v}|^2$. Substituting this in inequality (19) gives Cauchy's inequality (18).

Next we investigate a standard system of linear equations $A\vec{u} = \vec{v}$ where $A = (a_{ij})$ is an $n \times n$ matrix:

$$\begin{aligned} a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n &= v_1 \\ a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n &= v_2 \\ \vdots \qquad\quad \vdots \qquad\quad & \vdots \\ a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n &= v_n \end{aligned}$$

Then

$$\begin{aligned} |A\vec{u}|^2 = |\vec{v}|^2 &= v_1^2 + \cdots + v_n^2 \\ &= (a_{11}u_1 + \cdots + a_{1n}u_n)^2 + \cdots + (a_{n1}u_1 + \cdots + a_{nn}u_n)^2. \end{aligned}$$

Applying the Cauchy inequality to each of the terms on the last line above we find that

$$\begin{aligned} |A\vec{u}|^2 &\le \left(\sum_{j=1}^n a_{1j}^2 \right) |\vec{u}|^2 + \cdots + \left(\sum_{j=1}^n a_{nj}^2 \right) |\vec{u}|^2 \\ &= \left(\sum_{i,j=1}^n a_{ij}^2 \right) |\vec{u}|^2 \\ &= |A|^2 |\vec{u}|^2 \end{aligned},$$

where we defined $|A|^2 = \sum_{i,j=1}^n a_{ij}^2$ (this definition of $|A|$ is often called the *Frobenius norm* of A). Thus

$$|A\vec{u}| \le |A||\vec{u}|. \tag{20}$$

If the elements of matrix $A = a_{ij}(t)$ and the vector $\vec{u} = (v_1(t), \dots, v_n(t))$ are continuous functions of t for t in some interval $J \subset \mathbb{R}$, we measure the size of A and $\vec{u}$ over the whole interval J as follows:

$$\|A\|_J := \sup_{t \in J} |A(t)| \qquad \text{and} \qquad \|\vec{u}\|_J := \sup_{t \in J} |\vec{u}(t)|.$$

The inequality (20) thus implies

$$\|A\vec{u}\|_J \le \|A\|_J \|\vec{u}\|_J. \tag{21}$$

There is one more fact we will need about integrating a continuous vector-valued function $\vec{v}(t)$ on an interval $[a, b]$. It is the inequality

$$\left| \int_a^b \vec{v}(s)\, ds \right| \le \int_a^b |\vec{v}(s)|\, ds \tag{22}$$

To prove this, note that we define the integral of a vector-valued function $\vec{v}(t) = (u_1(t), \ldots, u_n(t))$ as integrating each component separately. Using this and the Cauchy inequality, we find that for any constant vector $\vec{V}$,

$$\langle \vec{V}, \int_a^b \vec{v}(s) \rangle \, ds = \int_a^b \langle \vec{V}, \vec{v}(s) \rangle \, ds \le |\vec{V}| \int_a^b |\vec{v}(s)| \, ds.$$

In particular, if we let let $\vec{V}$ be the constant vector $\vec{V} = \int_a^b \vec{v}(s) \, ds$, then

$$|\vec{V}|^2 = \langle \vec{V}, \vec{V} \rangle = \langle \vec{V}, \int_a^b \vec{v}(s) \rangle \, ds \le |\vec{V}| \int_a^b |\vec{v}(s)| \, ds.$$

After canceling $|\vec{V}|$ from both sides this is exactly the desired inequality (22).