Math 202, Fall 2013 Jerry L. Kazdan

Class Notes for Oct. 29, 2013

REMARK: Almost everything today will be an application of the Mean Value Theorem (or its special case, Rolle's Theorem. Please assume all functions have derivatives of any order you need.

Example 1. Say $f(0) = f(1) = f(2) = 0$. Show there is a point $c \in (0, 2)$ where $f''(c) = 0$.

SOLUTION. By Rolle's theorem there are points $c_1 \in (0, 1)$ and $c_2 \in (1, 2)$ where $f'(c_1) = 0$ and $f'(c_2) = 0$. Now apply Rolle's theorem again to the function $f'(x)$ in the interval $[c_1, c_2]$.

Example 2. Say $f(0) = f'(0) = 0$ and $f(2) = 0$. Show there is a point $c_2 \in (0, 2)$ where $f''(c_2) = 0$.

SOLUTION. By Rolle's theorem there is a point $c_1 \in (0, 2)$ where $f'(c_1) = 0$. Since $f'(0) = 0$ and $f'(c_1) = 0$, now apply Rolle's theorem again to $f'(x)$ in the interval $[0, c_1]$.

Example 3. Say $f(0) = f'(0) = f''(0$ and $f(2) = 0$. Show there is a point $c \in (0, 2)$ where $f'''(c) = 0$.

SOLUTION. By the previous Example there 1s a point $c_2 \in (0, c_1)$ where $f''(c_2) = 0$. Since $f''(0) = 0$ and $f''(c_2) = 0$, now apply Rolle's theorem again to $f''(x)$ in the interval $[0, c_2]$.

Example 4. Say $f(0) = 3$, $f(1) = 0$, and $f(3) = 2$. Show there is a point $c \in (0, 3)$ where $f''(c) = m$ for some $m > 0$. [Find an *explicit* value for m in terms of the given data for f

SOLUTION. The slope of the straight line joining $(0, 3)$ to $(1, 0)$ is -3. By the Mean Value Theorem there is a point $c_1 \in [0, 1]$ where $f'(c_1) = -3$.
Similarly, the slope of the straight line joining $(1, 0)$ to $(3, 2)$ is 1. By the Mean Value Theorem there is a point $c_2 \in [1, 3]$ where $f'(c_2) = 1$.
Now apply the Mean Value Theorem to $f'(x)$ in the interval $[c_1, c_2]$ to conclude there is a point $c \in [c_1, c_2]$ where $f''(c) = (1 - (-3))/(c_2 - c_1) = 4(c_2 - c_1) > 4/3$.

Example 5. A related example is if $a \le x \le b$ and $f(x)$ has the properties that $f(0) = f(b) = 0$, and $f''(x) \ge 0$ for all $x \in (a, b)$, then $f(x) \le 0$ for all $x \in [0, 1]$ (draw a picture!).

Proof by contradiction. Say that at some $\alpha \in (0, 1)$ $f(\alpha) > 0$. Then, reasoning as in the previous example, there is a $c \in (0, 1)$ so that $f''(c) < 0$, contradicting that $f''(x) \ge 0$.

Example 6. More generally: given $a < b < c$, say $f(a) > f(b)$ and $f(c) > f(b)$. Then there is an $m > 0$ so that $f''(\alpha) = m$ for some $\alpha \in (a, b)$.
One can prove this using the approach of the Example 4 (Exercise!). As an alternate, the following gives the best (that is, largest), value of m that works for *all* such functions f. Let $p(x)$ be the (unique) quadratic polynomial with the properties: $p(a) = f(a)$, $p(b) = f(b)$,

and $p(c) = f(c)$. Although not really needed here, Lagrange gave the elegant explicit formula

$$p(x) := f(a)\frac{(x-b)(x-c)}{(a-b)(a-c)} + f(b)\frac{(x-a)(x-c)}{(b-a)(b-c)} + f(c)\frac{(x-a)(x-b)}{(c-a)(c-b)}.$$

For simplicity, just write $p(x) = Ax^2 + Bx + C$ and let $g(x) = f(x) - p(x)$. Then $g(a) = g(b) = g(c) = 0$. Then by Rolle's Theorem applied twice (as in Example 1 above), there is an $\alpha \in (a, c)$ so that $g''(c) = 0$. But $p''(x) = 2A$ so $0 = f''(\alpha) - 2A$. That is, $f''(\alpha) = 2A$. In the case of the data given here one finds that $A > 0$. It is the best possible value since that is what one gets in the special case where $f(x) \equiv g(x)$.

Example 7. Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that the first derivative exists for all x but the first derivative is *not* continuous at $x = 0$.

SOLUTION. By the usual rules (particularly the chain rule), for all $x \neq 0$, this function is clearly differentiable and its derivative is

$$f'(x) = 2x\sin(1/x) - \cos(1/x), \qquad x \neq 0,$$

while at $x = 0$,

$$\lim_{h\to 0}\left(\frac{h^2\sin(1/h)}{h}\right) = 0.$$

Thus, f is differentiable at $x = 0$ and $f'(0) = 0$.
Now notice that because of the oscillating term $\cos(1/x)$, as $x \to 0$, the derivative, $f'(x)$, has no limiting value. Thus $f'(x)$ is not continuous at $x = 0$.

Taylor Polynomial. [This is discussed in essentially all calculus texts.]

Problem: Given a function $f(x)$, find a cubic polynomial, $p_3(x) := a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3$, that agrees closely with f at $x = c$ in the sense that $f(c) = p(c)$, $f'(c) = p'(c)$, $f''(c) = p''(c)$, and $f'''(c) = p'''(c)$.

SOLUTION: Clearly $p(c) = a_0$, $p'(c) = a_1$, $p''(c) = 2!a_2$, and $p'''(c) = 3\cdot 2a_3$. Thus, to match the conditions we need

$$a_0 = f(c) \qquad a_1 = f'(c), \qquad a_2 = \frac{f''(c)}{2!}, \qquad a_3 = \frac{f'''(c)}{3!}.$$

This polynomial

$$p_3(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)(x-c)^3}{3!}. \tag{1}$$

is called the *Taylor polynomial of f of order 3 at $x = c$*.

We would like a formula for the error in using the approximation (1) at a specific point $x \neq c$. It uses the Mean Value Theorem.

$$\text{Error}_3(x) := f(x) - p_3(x) = f(x) - \left[f(c) + f'(c)(x-3) + \frac{f''(c)}{2!}(x-3)^2 + \frac{f'''(c)}{3!}(x-c)^3\right].$$

Since we are thinking of x as being a specific point, we introduce a new variable t and let

$$\begin{aligned} Q_3(t) :=& \text{Error}_3(t) - K(t-c)^4 = f(t) - \left[p_3(t) + K(t-c)^4\right] \\ =& f(t) - \left[f(c) + f'(c)(t-c) + \frac{f''(c)}{2!}(t-c)^2 + \frac{f'''(c)}{3!}(t-c)^3 + K(t-c)^4\right], \end{aligned} \tag{2}$$

and pick K so that $Q_3(x) = 0$. Now notice that

$$Q_3(c) = Q_3'(c) = Q_3''(c) = Q_3'''(c) = 0 \qquad \text{by our choice of } \ p_3(x)$$

and also $Q_3(x) = 0$ by our choice of K. By the same reasoning as Example 3 above, there is a point α between c and x where the fourth derivative, $Q_3^{(4)}(\alpha) = 0$. But by a direct calculation, $Q_3^{(4)}(t) = f^{(4)}(t) - 4!K$. Letting $t = \alpha$ we find that $K = f^{(4)}(\alpha)/(4!)$. Now use this in equation (2) and use the specific point $t = x$ (recall that $Q_3(x) = 0$) to find

$$\begin{aligned} 0 = Q_3(x) =& \text{Error}\,_3(x) - \frac{f^{(4)}(\alpha)}{4!}(x-c)^4 = f(x) - \left[p_3(x) + \frac{f^{(4)}(\alpha)}{4!}(x-c)^4\right] \\ =& f(x)) - \left[f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \frac{f^{(4)}(\alpha)}{4!}(x-c)^4\right], \end{aligned}$$

In other words, we have Taylor's Theorem of order 3 centered at $x = c$

$$f(x)) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \frac{f^{(4)}(\alpha)}{4!}(x-c)^4 \tag{3}$$

The identical reasoning gives Taylor's Theorem of order n centered at $x = c$: there is an α between c and x so that

$$f(x) = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \frac{f^{(n+1)}(\alpha)}{(n+1)!}(x-c)^{n+1} \tag{4}$$

In many ways, the Taylor polynomial for f near $x = c$ is like looking at f through a microscope. Taking more terms is like using a microscope with greater magnification. The Mean Value Theorem is the case $n = 0$.

Second derivative test for a local minimum. This is an easy application of Taylor's Theorem of order 2. Say $x = c$ is a *critical point* of $f(x)$, that is, $f'(c) = 0$. We would like to test if this is a local max or min. We show that if $f''(c) > 0$ it is a local minimum. Since $f'(c) = 0$, by Taylor's Theorem of order 2 at $x = c$, there is an α between c and x so that

$$f(x) - f(c) = \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(\alpha)}{3!}(x-c)^3.$$

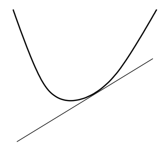
Because $f''(c) > 0$ and for x near c the term $(x-c)^3$ is much smaller than $(x-c)^2$, we see that $f(x) - f(c) > 0$ for x sufficiently close to c. Therefore f has a local minimum at $x = c$.

Similarly, if $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.

If both $f'(c) = 0$ and $f''(c) = 0$ this test fails since the determination of a local max or min will involve the higher order terms in the Taylor polynomial. The following simple examples make this clearer.

EXAMPLES 8. The functions x^3, x^4, and $-x^4$ all have their first and second derivatives zero at the origin. Draw a sketch of their graphs (please). Of these examples, x^3 has neither a local man or min there, x^4 has a local min at the origin, while $-x^4$ has a local max at the origin.

EXAMPLE 9. If $f''(x) \geq 0$ everywhere, then at any point c the graph of $f(x)$ lies above its tangent line.

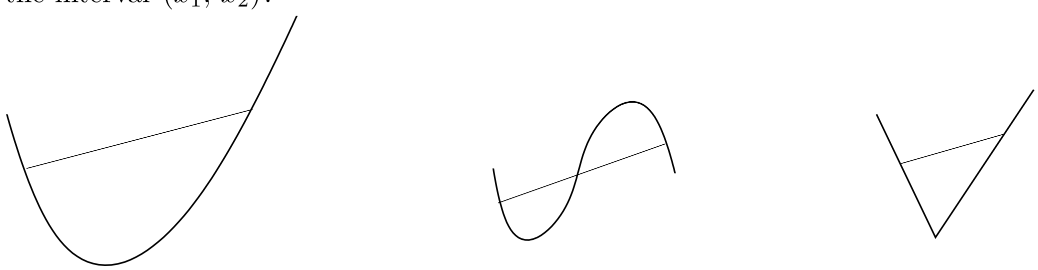


PROOF: The equation of the tangent line is $g(x) = f(c) + f'(c)(x - c)$. By Taylor's Theorem of order 1, there is an α between c and x

$$f(x) = f(c) + f'(c)(x - c) + \frac{1}{2} f''(\alpha)(x - c)^2.$$

But the first part of the right-hand side is the equation of the tangent line and $f''(\alpha) \geq 0$. Therefore $f(x) \geq g(x)$ for all x.

Convex functions A function $f(x)$ is *convex* in an interval $[a, b]$ if for every pair of points x_1, x_2 in $[a, b]$ the chord between $(x_1, f(x_1))$ and $(x_2, f(x_2))$ lies above the graph of f in the interval (x_1, x_2).



convex not convex convex

Since the equation of the chord is $y = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$, convexity in $[a, b]$ means

$$f(x) \leq f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) \quad \text{for all} \quad x_1, x_2 \in [a, b]. \tag{5}$$

There is a useful alternate way to write this. Every point $x \in [x_1, x_2]$ can be written in the form $x = (1 - t)x_1 + tx_2$ for some $0 \leq t \leq 1$. This gives $t = (x - x_1)/((x_2 - x_1)$ Thus equation (5) can be rewritten as

$$f((1 - t)x_1 + tx_2) \leq f(x_1) + (f(x_2) - f(x_1))t,$$

that is, for any x_1 and x_2 $(x_1 \neq x_2)$ in $[a, b]$,

$$f((1 - t)x_1 + tx_2) \leq (1 - t)f(x_1) + tf(x_2) \qquad \text{for all} \quad t \in [0, 1]. \tag{6}$$

Since $|x|$ is a convex function, a convex function need not be differentiable everywhere. However

Theorem If f is twice differentiable in $[a, b]$ and $f''(x) \geq 0$, then f is convex there.

We prove that inequality (5) holds. First we reduce to the special case where the chord is horizontal and on the x-axis by letting

$$h(x) := f(x) - \left[f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) \right].$$

Then $h(x_1) = h(x_2) = 0$ and $h''(x) = f''(x) \geq 0$. Proving that inequality (5) holds means proving that $h(x) \leq 0$ in $[x_1, x_2]$. But this was proved in Example 5.

[Last revised: November 8, 2013]