Math 202, Fall 2013 Jerry L. Kazdan

Problem Set 9

DUE: In class Tuesday, Nov. 12. *Late papers will be accepted until 1:00 PM Wednesday.*

REMARK: Please re-read Chapter 16, pages on Differentiation.

PROBLEMS

1. Prove that two differentiable functions on the interval (0, 1) have the same derivative if and only if they differ by a constant.

2. We defined the function e^x by its power series. One can use this to show the fundamental property $e^{x+y} = e^x e^y$. Note this implies that $e^x = (e^{x/2})^2 > 0$
   a) Assuming the above, show that e^x is differentiable for all real x and $d(e^x)/dx = e^x$.
   b) Show that $e^x \ge 1 + x$ for all (real) x. [SUGGESTION: Use $(e^x)'' = e^x > 0$.]
   c) If for some constant γ the differentiable function $v(x)$ satisfies $v' - \gamma v \le 0$, showw that $v(x) \le v(0)e^{\gamma x}$ for all $x \ge 0$. [HINT: Consider $g(x) := e^{-\gamma x}v(x)$.]

3. Say a function $f(x)$ has the property that there is a constant $\alpha > 0$ and a constant $C > 0$ so that
$$|f(x_2) - f(x_1)| \le C|x_2 - x_1|^{1+\alpha} \qquad \text{for all real} \quad x_1, x_2.$$
   Show that $f(x) \equiv$ constant.
   Also give an example of a *non-constant* function that does satisfy this with $\alpha = 0$.

4. Let $f(x) : \mathbb{R} \to \mathbb{R}$ be a bounded smooth function for $-\infty < x < \infty$. Show that $f''(c) = 0$ for at least one point c. Thus, f has at least one inflection point.

5. Let $p(x) := (x-1)(x-2)\cdots(x-6) = x^6 - 21x^5 + \cdots$ and let $p(x, \epsilon)$ be the polynomial obtained by replacing $-21x^5$ by $-(21+\epsilon)x^5$. Let $x(\epsilon)$ denote the perturbed value of root $x = 4$, so $x(0) = 4$. Compute the sensitivity of this root as one changes ϵ, that is, compute $dx(\epsilon)/d\epsilon\big|_{\epsilon=0}$.

6. a) Let x_j, $j = 1, 2, \ldots$ be a sequence of points in $\mathbb{R}$. If $|x_{j+1} - x_j| \le \dfrac{1}{j^4}$, show that these points converge. [HINT: Cauchy sequence]
   b) A generalization. Let $\{x_j\}$ be a sequence of points in $\mathbb{R}$ with the property that
$$\sum_j |x_{j+1} - x_j| < \infty.$$
   Prove that the sequence $\{x_j\}$ converges.
   Give an example of a convergent sequence that does not have this property.

Bonus Problems

[Please give your solutions directly to Professor Kazdan]

1-B This problem concerns a classical example of a function $f(x) \not\equiv 0$ that has derivatives of all order and all of whose derivatives are zero at the origin are zero. Thus its Taylor polynomials at the origin are all zero. While at first this seems like pathology, this function is quite useful.

a) Let $x > 0$. Show that $\lim_{x\to 0} \dfrac{1}{x^k} e^{-1/x} = 0$ for any integer k.

b) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{for } x > 0, \\ 0 & \text{for } x \le 0, \end{cases}.$$

Show that f is smooth, that is, $f(x)$ and all of its derivatives exist and are continuous for all real x. Sketch the graph.

c) Show that each of the following are smooth and sketch their graphs:

$$g(x) = f(x)f(1-x), \qquad h(x) = \frac{f(x)}{f(x) + f(1-x)},$$

$$k(x) = h(x)h(4-x), \qquad K(x) = k(x+2), \qquad H(x) = K(4x)$$

2-B Say a function $u(x)$ satisfies the differential equation

$$u'' + b(x)u' + c(x)u = 0 \tag{1}$$

on the interval $[0, A]$ and that the coefficients $b(x)$ and $c(x)$ are both bounded, say $|b(x)| \le M$ and $|c(x)| \le M$ (if the coefficients are continuous, this is always true for some M).

a) Define $E(x) := \frac{1}{2}(u'^2 + u^2)$. Show that for some constant γ (depending on M) we have $E'(x) \le \gamma E(x)$. [Suggestion; use the inequality $2ab \le a^2 + b^2$.]

b) Use Problem 2(c) above to show that $E(x) \le e^{\gamma x} E(0)$ for all $x \in [0, A]$.

c) In particular, if $u(0) = 0$ and $u'(0) = 0$, show that $E(x) = 0$ and hence $u(x) = 0$ for all $x \in [0, A]$. In other words, if $u'' + b(x)u' + c(x)u = 0$ on the interval $[0, A]$ and that the functions $b(x)$ and $c(x)$ are both bounded, and if $u(0) = 0$ and $u'(0) - 0$, then the only possibility is that $u(x) \equiv 0$ for all $x \ge 0$.

d) Use thuis to prove the *uniqueness theorem*: if $v(x)$ and $w(x)$ both satisfy equation (1) and have the same initial conditions, $v(0) = w(0)$ and $v'(0) = w'(0)$, then $v(x) \equiv w(x)$ in the interval $[0, A]$.

3-B [Interpolation] Say $f(x)$ is a smooth function on the interval $[a, b]$ and you approximate it by the chord

$$p(x) := f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

that passes through the end points $(a, f(a))$ and $(b, f(b))$. We seek an estimate for the error, $f(x) - p(x)$.

Fix this specific point x ($x \neq a$ and $x \neq b$) and let

$$E(t) := f(t) - [p(t) + K(t - a)(t - b)],$$

where the constant K is chosen so that $E(x) = 0$. Show that $K = \frac{1}{2}f''(c)$ for some point $c \in (a, b)$ and conclude that

$$\begin{aligned} f(x) =& p(x) + \frac{1}{2}f''(c)(x - a)(x - b) \\ =& f(a) + \frac{f(b) - f(a)}{b - a}(x - a) + \frac{1}{2}f''(c)(x - a)(x - b) \end{aligned}$$

This also shows that if $f \geq 0$ then for x in the interval $[a, b]$, the graph of $f(x)$ lies below the chord joining the points $(a, f(a))$ and $(b, f(b))$.

[Last revised: November 12, 2013]