Math 202, Fall 2013 Jerry L. Kazdan

Problem Set 7

Due: In class Thursday, Oct. 24. *Late papers will be accepted until 1:00 PM Friday.*

Remarks:

Please re-read Chapter 15 on Continuity and read Chapter 16, pages 307-317 on Differentiation.

As always, in a number of the problems, the issue is to find the key idea and state it clearly. And then to have a well organized explanation that is easy to read and understand.
It is far more important to concentrate on this than on the kind of formal proofs (statement, reason, statement, reason, etc.) that we had at the beginning.
...And to have fun with the problems.

Problems

1. In class, if $z = x + iy$ we defined e^z by a power series and observed that, $e^{iz} = \cos z + i\sin z$. In particular, for real x, $e^{ix} = \cos x + i\sin x$. From the power series one cam also show that for any complex z and w the usual formula $e^{z+w} = e^z e^w$ remains valid. Use the observation that

   $$1 + \cos x + \cos 2x + \cdots + \cos nx = \text{Real part of } \{1 + e^{ix} + e^{2ix} + \cdots + e^{nix}\}$$

   and that the right hand side is a geometric series to find a formula for $1 + \cos x + \cos 2x + \cdots + \cos nx$. [Assume x is not a multiple of 2π]. Your resulting formula should not have any complex numbers.

2. [#15.3] [T/F] If $f : \mathbb{R} \to \mathbb{R}$ is continuous everywhere and $f(x) = 0$ for all rational numbers x, then $f(x) = 0$ for all real x.

3. [#15.5] [T/F] The function $f(x) := |x|^3$ is continuous for all $x \in \mathbb{R}$.

4. [#15.7] [T/F] Let f, g, and h be continuous on the interval $[0, 2]$. If $f(0) < g(0) < h(0)$ and $f(2) > g(2) > h(2)$, then there exists some $c \in [0, 2]$ such that $f(c) = g(c) = h(c)$.

5. [#15.10][T/F]
   a) If f is continuous on $\mathbb{R}$, then f is bounded.
   b) If f is continuous on $[0, 1]$, then f is bounded.
   c) If f is continuous on $\mathbb{R}$ and is bounded, then f attains its supremum.

6. [#15.12] Construct a function f with the property that there are sequences a_n and b_n converging to zero such that $f(a_n)$ converges to zero but $f(b_n)$ is unbounded.

   Does there exist such a function f that is continuous at $x = 0$?

7. [#15.15] Let $f(x) := x^2 + 4x$. Clearly $\lim_{x\to 0} f(x) = 0$. Assuming that $0 < \epsilon < 4$, how small must δ be so that $|x| < \delta$ implies that $|f(x)| < \epsilon$? Express δ as a function of ϵ.

8. [#15.17] Let $f(a, n) := (1 + a)^n$, where a and n are positive.
   a) For constant a, how does $f(a, n)$ behave as $n \to \infty$? For constant n, how does $f(a, n)$ behave as $a \to 0$?
   b) Let $L \geq 1$ be a given real number. Prove that there exists a sequence $a_n \to 0$ and $f(a_n, n) \to L$ as $n \to \infty$. In other words, depending on the choice of a_n, f may approach any value.

9. [#15.21] Prove that there exists $x \in [1, 2]$ such that $x^5 + 2x + 5 = x^4 + 10$.

10. Given any real number $c > 0$, prove there is an $x > 0$ such that $x^{17} + 8x^2 = c$.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous (real-valued) function that is continuous at $x = a$. If $f(a) > 0$, show there is an interval $J := \{x \in \mathbb{R} \,|\, |x - a| < \delta\}$ so that if $x \in J$, then $f(x) > f(a)/2$.

12. [#15.24] Prove that any (real) polynomial whose degree is odd must have at least one real root.

Bonus Problem

[Please give your solutions directly to Professor Kazdan]

1-B [#15.33] Let n be a positive integer, and suppose f is continuous on the interval $[0, 1]$ with $f(0) = f(1)$. Prove that the graph of f has a horizontal chord of length $1/n$. In other words, prove that there is an $x \in [0, \frac{n-1}{n}]$ such that $f(x + \frac{1}{n}) = f(x)$.

2-B In class (and in the book: page 275, 14.11) we proved that if a real sequence a_n converges, then so does the sequence of averages:

$$S_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Although the sequence $a_n = (-1)^n$ does not converge, its sequence of averages converges (to zero). If a sequence of averages S_n converges, must the original sequence a_n be *bounded*? Proof or counterexample.

[Last revised: October 18, 2013]