Math 202, Fall 2013 Jerry L. Kazdan

Problem Set 6

DUE: In class Thursday, Oct. 17. *Late papers will be accepted until 1:00 PM Friday.*

REMARKS:
Please read Chapter 15 pages 302 - 304 on Continuity.

In doing the proofs, it is no longer necessary to justify each arithmetic statement as we did in the beginning of the course. Just do common sense arithmetic and compute simple limits when needed.
As always, in a number of the problems, the issue is to find the key idea and state it clearly. And then to have a well organized explanation that is easy to read and understand.
It is far more important to concentrate on this than on the kind of formal proofs (statement, reason, statement, reason, etc.) that we had at the beginning.
...And to have fun with the problems.

PROBLEMS

1. [#14.32] A runaway train is hurtling toward a brick wall at a speed of 100 miles per hour. When it is 2 miles from the wall, a (speedy) fly begins to fly repeatedly between the train and the wall at the speed of 200 miles per hour. Determine how far the fly travels before it is smashed.

2. [#14.30] Let x_n be the sequence defined recursively by $x_1 = 1$ and $x_{n+1} = 1/(x_1 + \cdots + x_n)$. Prove that this sequence converges and obtain the limit.

3. [#14.33] If $\sum_{k=1}^{\infty} a_k$ converges to A and $\sum_{k=1}^{\infty} b_k$ converges to B, show that $\sum_{k=1}^{\infty}(a_k + b_k)$ converges to $A + B$.

4. [#14.36] Find the rational number whose repeating decimal expansion is $.247247247\ldots$.

5. Show that $n^{\frac{1}{n}} \to 1$. [HINT: Let $x_n := n^{\frac{1}{n}} - 1$. Show that $x_n \to 0$ similarly to HW 4, Problem 5, by proving and using the inequality: if $a > 0$, then $(1+a)^n > 1 + na + \frac{n(n-1)}{2}a^2$].

6. Show that the power series $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}$ converges for all $z \in \mathbb{C}$.

   [REMARK: In calculus, for real $z = x$ this was found to be the Taylor series for $\cos x$. For complex z, this convergent series is used to *define* $\cos z$].

7. In class – and in the book – we showed that the *harmonic series* diverges to infinity. Find an integer N so that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} > 100.$$

8. [#15.2] There is a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ if and only if $x \in \mathbb{Z}$. Why or why not?

9. [#15.4] There is an $x \in \mathbb{R}$ so that $\frac{x^2+5}{3+x^7} = 1$. Why or why not?

10. [15.8] let $f : \mathbb{R} \to \mathbb{R}$. If $|f|$ is continuous, then f is continuous. Proof or counterexample.

Bonus Problems

[Please give your solutions directly to Professor Kazdan]

1-B The number e is defined as

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots .$$

Prove that e is *not* a rational number by the following steps.

a) Show that $2 < e < 3$. So e is definitely not an integer.

b) By contradiction, say $e = \frac{p}{q}$, where p and q are positive integers with $q \geq 2$. Show that

$$e\,q! = N + \frac{c}{q+1},$$

where N is an integer and $0 < c < e$. Thus, conclude that $\frac{c}{q+1}$ must be an integer.

c) Then show that this contradicts $e < 3$ and $q + 1 \geq 3$.

2-B Let $A = [a_{ij}]$ be an $n \times n$ real matrix. Define its *norm* by the rule

$$|A| := \sqrt{\sum_{i,j=1}^{n} a_{ij}^2}.$$

You may use that the triangle inequality holds: $|A + B| \leq |A| + |B|$.

a) If A and B are $n \times n$ matrices, show that $|AB| \le |A||B|$.
SUGGESTION: Prove that $|AB|^2 \le |A|^2|B|^2$ using the *Cauchy inequality* (p. 23 #30):

$$\left[\sum_{j=1}^{n} |x_j y_j|\right]^2 \le \left[\sum_{j=1}^{n} x_j^2\right]\left[\sum_{j=1}^{n} y_j^2\right]$$

applied to $\sum_{j=1}^{n} a_{ij} b_{jk}$. [One consequence: $|A^k| \le |A|^k$ for any positive integer k.]

b) If A_j, $j = 1, 2, 3, \ldots$, and L are $n \times n$ matrix, we say that A_j *converges to* L if $|A_j - L| \to 0$. If $A_j \to L$, show that the sequence $|A_j|$ is bounded.

c) If also B_j are a sequence of $n \times n$ matrices and $B_j \to M$, show that

$$A_j B_j \to LM.$$

[Last revised: October 8, 2018]