

Problem Set 5

DUE: In class Tuesday, Oct. 8. *Late papers will be accepted until 1:00 PM* Wednesday.

Please read Chapter 14 and Chapter 15, pages 293 - 301 (about Continuity).

As always, in a number of the problems, the issue is to find the key idea and state it clearly. And then to have a well organized explanation that is easy to read and understand.

It's more important to concentrate on this than on the kind of formal proofs (statement, reason, statement, reason, etc.) that we had at the beginning.

Have fun with the problems.

Jerry Kazdan

PROBLEMS

1. Let a_n be a real sequence that converges to L and b_n be a real sequence that converges to M . Define a new sequence $c_n := \max(a_n, b_n)$. Either prove that this sequence converges or give a counterexample.
2. [#13.27] Let $a_n = \sqrt{n^2 + n} - n$. Show that it converges and compute the limit.
3. A sequence $x_n \in \mathbb{R}$ is called *contracting* if for some constant $0 < c < 1$ (such as $c = \frac{1}{2}$) it has the property that for all $n = 1, 2, 3, \dots$

$$|x_{n+1} - x_n| \leq c|x_n - x_{n-1}|.$$

The point of this problem is to show that a contracting sequence converges.

- a) Show that $|x_{n+1} - x_n| \leq c^n|x_1 - x_0|$ for all n .
- b) Use $x_{n+1} - x_0 = (x_{n+1} - x_n) + (x_n - x_{n-1}) + \dots + (x_1 - x_0)$ to show that

$$|x_{n+1} - x_0| \leq (c^n + c^{n-1} + \dots + c + 1)|x_1 - x_0|$$

- c) More generally, if $n > k$ show that

$$\begin{aligned} |x_{n+1} - x_k| &\leq (c^n + c^{n-1} + \dots + c^k)|x_1 - x_0| \\ &= c^k \left(\frac{1 - c^{n-k+1}}{1 - c} \right) |x_1 - x_0| < c^k \frac{|x_1 - x_0|}{1 - c}. \end{aligned}$$

- d) Show that the x_n are a Cauchy sequence — and hence converge.

4. Let $a_1 = 1$ and define the a_n , $n \geq 2$, recursively by the rule $a_{n+1} := \sqrt{1 + a_n}$. Prove this sequence converges. Then find the value of the limit.

5. If $0 < p < 1$, show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges. [We did the case $p = 1/2$ in class.]
6. [#14.50] Determine whether $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$ converges.
7. [#14.56, *Ratio Test for divergence*]. Say we have a (possibly complex) sequence such that $|a_{n+1}/a_n| \rightarrow \rho$ for some $\rho > 1$. Show that the series $\sum_{n=1}^{\infty} a_n$ diverges.
8. Use the ratio test to determine the center and radius of the disk of convergence (in the complex plane) for the following power series. (Don't worry about the points on the boundary of the disk).

$$a). \sum_{n=1}^{\infty} n^2 z^n, \quad b). \sum_{n=0}^{\infty} \frac{z^n}{1+2^n}, \quad c). \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \quad d). \sum_{n=0}^{\infty} \frac{(z-2i)^n}{n^2}$$

9. Consider the *alternating series* $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$ and let S_n be the sum of the first n terms.
- a) Show that the even terms, S_2, S_4, \dots, S_{2n} are increasing and the odd terms, $S_1, S_3, \dots, S_{2n+1}$, are decreasing.
- b) Show that $S_{2n+1} - S_{2n} = 1/(2n+1) > 0$.
- c) Show that the even terms S_{2n} converge to some number α and the odd terms S_{2n+1} converge to some number β .
- d) Show that $\alpha = \beta$ and conclude that the whole series converges.

10. [#14.60] Which of the following series converge and which diverge? Why?

$$a). \sum_{n=1}^{\infty} \frac{2n^2 + 15n + 2}{n^4 + 3n + 1}, \quad b). \sum_{n=1}^{\infty} \frac{2n^2 + 15n + 2}{n^3 + 3n + 1}, \quad c). \sum_{n=1}^{\infty} \frac{3 + 5n + n^2}{2^n}.$$

11. a) Let \mathbb{N}_k be the set of the first k positive integers $1, \dots, k$ and L the set of positive integers *not* divisible by 2. Define the sequence

$$S_k := \frac{\text{number of integers in the set } L \cap \mathbb{N}_k}{k}.$$

Compute $\lim_{k \rightarrow \infty} S_k$.

- b) Let M the set of positive integers *not* divisible by 2 or 3. Define the sequence

$$T_k := \frac{\text{number of integers in the set } M \cap \mathbb{N}_k}{k}.$$

Compute $\lim_{k \rightarrow \infty} T_k$.