

Problem Set 4

DUE: In class Thursday, Sept. 26. *Late papers will be accepted until 1:00 PM Friday.*

REPEAT OF LAST WEEK'S REMARKS:

In doing the proofs, it is no longer necessary to justify each arithmetic statement as we did in the beginning of the course. Just do common sense arithmetic when needed.

Only exception is careful use of the Least Upper Bound axiom.

In a number of the problems, the issue is to find the key idea and state it clearly. And then to have a well organized explanation that is easy to read and understand.

It's more important to concentrate on this than on the kind of formal proofs (statement, reason, statement, reason, etc.) that we had at the beginning.

Lots of problems this week. Fortunately a number of them are short – but don't wait until Wednesday night!

Enjoy the weekend . . . and have fun with the problems.

Jerry Kazdan

1. Please reread Chapter 14, pages 271 - 279.
2. Please read Chapter 18, Sections 18.1 - 18.7.
3. Note that Exam 1 will be on Tuesday, October 1 in class from 12:00-1:20. Closed book but you may use one 3×5 card with notes on both sides. It will cover Chapters 1, 13, 14 (pages 271-279), and 18 (Sections 18.1 - 18.7) from the book.

PROBLEMS

1. A tennis ball is dropped from a height H . After each bounce it returns to two-thirds of its height on the previous bounce. How far does the ball travel until it is at rest on the floor?
2. [# 13.24] let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be bounded functions such that $f(x) \leq g(x)$ for all x . Let F denote the image of f and G the image of g . Give examples (with pictures) of pairs of such functions with:

$$\text{a). } \sup(F) < \inf(G) \quad \text{b). } \sup(F) = \inf(G) \quad \text{c). } \sup(F) > \inf(G)$$

3. [# 13.30] Let $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$. Show that $\lim_{n \rightarrow \infty} x_n$ exists. [REMARK: In fact, the limit equals $\ln 2$ but that is not needed for this exercise.]
4. [# 13.32] *Nested Interval Property*. Let $\{I_n \subset \mathbb{R}\}$ be a sequence of closed (non-empty) intervals with I_n having length d_n such that $I_{n+1} \subseteq I_n$ and $d_n \rightarrow 0$. The Nested Interval Property states that for such a sequence there is exactly one point that belongs to all of the I_n .

- a) Show that our Completeness Axiom implies the Nested Interval Property.
- b) Show that the Nested Interval Property implies our Completeness Axiom.
5. [#14.22] If $c > 0$, show that $\lim_{n \rightarrow \infty} c^{\frac{1}{n}} = 1$. [SUGGESTION: The case $c = 1$ is clear. If $c > 1$, let $x_n := c^{\frac{1}{n}} - 1$ and show that $x_n \rightarrow 0$. Note $x_n > 0$ so use $c = (1 + x_n)^n \geq 1 + nx_n$. If $0 < c < 1$, take reciprocals.]
6. [#14.2] For each condition below, give an example of an *unbounded* sequence such that $a_{n+1} - a_n > 0$ for all $n \in \mathbb{N}$ and the specified condition holds.
- a) $\lim (a_{n+1} - a_n) = 0$.
- b) $\lim (a_{n+1} - a_n)$ does not exist.
- c) $\lim (a_{n+1} - a_n) = L$, where $L > 0$.
7. Suppose that $x_0 = c$ for some real c and $x_{n+1} = \sqrt{1 + x_n^2}$ for all $n \in \mathbb{N}$. For which c does x_n converge? Why?
8. [#14.9]. Proof or counterexample. Suppose that $x_n \rightarrow L$.
- a) For all $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $|x_{n+1} - x_n| < \epsilon$.
- b) There exists $n \in \mathbb{N}$ such that for all $\epsilon > 0$, $|x_{n+1} - x_n| < \epsilon$.
- c) There exists $\epsilon > 0$ such that for all $n \in \mathbb{N}$: $|x_{n+1} - x_n| < \epsilon$.
- d) For all $n \in \mathbb{N}$ there exists $\epsilon > 0$ such that $|x_{n+1} - x_n| < \epsilon$.
9. Let s_n be a sequence of real numbers that converge to some $S > 0$. Show there is an integer $N > 0$ so that if $n \geq N$ then $s_n > S/2$.
10. [#14.14] Let a_n and $b_n \neq 0$ be real sequences. If $a_n \rightarrow L$ and $b_n \rightarrow M \neq 0$, show that $a_n/b_n \rightarrow L/M$.
- SUGGESTION: First do the special case where all the $a_n = 1$, the $b_n > 0$, and $M > 0$. [Where in your proof did you use that $M > 0$?]. Then use Theorem 14.5b to get the general case.
11. [#14.15] Let b and L be real numbers. If $b \leq L + \epsilon$ for all $\epsilon > 0$, prove that $b \leq L$.
12. [#14.18] If $a_1 = 1$ and $a_{n+1} = \sqrt{3a_n + 4}$ for $n \geq 1$, show that $a_n < 4$ for all $n \geq 1$.
13. [#14.19]. Suppose that $x_1 = 1$ and $2x_{n+1} = x_n + 3/x_n$ for $n \geq 1$. Prove that $\lim_{n \rightarrow \infty} x_n$ exists – and find the limit.

14. [#18.6] If w_1 and w_2 are distinct points in \mathbb{C} . give a geometric description of the set $\{z \in \mathbb{C} : |z - w_1| = |z - w_2|\}$.
15. [#18.7] Prove the following properties of complex conjugation for all complex numbers z and w :
- a). $\overline{z\bar{w}} = \bar{z}w$ b). $\overline{z + w} = \bar{z} + \bar{w}$ c). $|\bar{z}| = |z|$
16. If c is a complex number with $|c| < 1$, show that $(n^2 + 1)c^n \rightarrow 0$. Does $n^5 c^n$ converge? If so, to what? Explain your reasoning. [SUGGESTION: Prove (and use) that a complex sequence z_n converges to zero if and only if the real sequence $|z_n|$ converges to zero.]

[Last revised: October 2, 2013]