Math 202, Fall 2013 Jerry L. Kazdan

Problem Set 2

DUE: In class Thursday, Sept. 12. *Late papers will be accepted until 1:00 PM Friday.*

REMARKS:

Sometimes the statement of a problem requires you to interpret what the author is asking for. As an example, in problem 3 below, you are asked to “determine” a certain set of numbers, and it is not clear what the command “determine” means. The intent of the author is that you should make clear which values of x satisfy the given inequality, for example by expressing the set of such x as a union of intervals. If you do that, remember to indicate whether the interval contains its endpoints, using symbols [and] , or whether the endpoints are omitted, using symbols (and) . For instance you can write the interval $-2 \le x < 3$ as $[-2, 3)$.

Start right away on the problems, doing those you can first, and keeping in mind some problem which is giving you difficulty. Then come back to that problem later the same day, or the next day. This is the way to maximize your receptivity to good ideas and inspiration.

Please allow 15 hours per week, outside of class, for this course, for reading the text, working on homework alone and with friends, and for thinking about hard problems.
You won’t always need this much time, but sometimes you will.
Remember if you work with friends on the homework to get started on it yourself, then work with them in the middle, and then always write up your solutions by yourself.

In our textbook, reread Chapter 1 and also read Chapter 13 before class on Tuesday.

NOTES

Many in the class seemed to be confused in using the properties of a field (Section 1.39) to deduce other properties (Section 1.43) since all of the statements seem so obvious that you use them subconsciously. The following artificial notation helped me. Instead of writing the additive inverse of x as $-x$, try writing it as $\hat{x}$. Thus for any x in a field, we have $x+\hat{x}=0$. By the commutativity of addition, $\hat{x}+x=0$. Thus the additive inverse of $\hat{x}$ is x, that is, $\hat{\hat{x}}=x$. Also, we *define subtraction* by $x-y=x+\hat{y}$. Using this notation here are two examples that we did in class.

1.43a PROOF THAT IF $x+z=y+z$, THEN $x=y$
Add $\hat{z}$ to both sides: $(x+z)+\hat{z}=(y+z)+\hat{z}$. But $(x+z)+\hat{z}=x+(z+\hat{z})=x$ and similarly $(y+z)+\hat{z}=y$. Thus $x=y$.

1.43b PROOF THAT $x\cdot 0=0$.
Since $0+0=0$, then $x\cdot 0=x\cdot(0+0)=x\cdot 0+x\cdot 0$. Now add $(\widehat{x\cdot 0})$ to both sides to find that $0=x\cdot 0$.

1.43c PROOF THAT $(-x)y=-(xy)$. That is, $\hat{x}y=\widehat{(xy)}$.
Adding xy to both sides (as in 1.43a) this is equivalent to $\hat{x}y+xy=0$. But $\hat{x}y+xy=(\hat{x}+x)y=0y=0$ (by 1.43b).

Problems

1. [Sec. 1.43] Prove parts (d)-(g):
   - (d). $\hat{x} = (\hat{1})x$
   - (e). $(\hat{x})(\hat{y}) = xy$
   - (f). If $xz = yz$ and $z \neq 0$, then $x = y$
   - (g). $xy = 0$ implies $x = 0$ or $y = 0$

2. [Sec. 1.46]. (You may use Sec. 1.45) In an ordered field:
   - (a). $x \leq y$ implies $-y \leq -x$
   - (b). If $x \leq y$ and $z \leq 0$, then $yz \leq xz$
   - (c). If $0 \leq x$ and $0 \leq y$ then $0 \leq xy$
   - (d). $x^2 \geq 0$ for any x
   - (e). $0 < 1$
   - (f). $0 < x$ implies $0 < x^{-1}$
   - (g). $0 < x < y$ implies $0 < y^{-1} < x^{-1}$

3. [p. 22 #27] Determine the set of real solutions to $|x/(x+1)| \leq 1$.

4. [p. 22 #29] Let x, y, and z be non-negative real numbers such that $y + z \geq 2$. Prove that $(x+y+z)^2 \geq 4x + 4yz$ and determine when equality holds.

5. [p. 23 #30] Let x. y, u, and v be real numbers.
   - a) Prove that $(xu + yv)^2 \leq (x^2 + y^2)(u^2 + v^2)$.
   - b) Determine precisely when equality holds in part a).

6. [p. 23 #31] *Extensions of the arithmetic-geometric mean inequality.*
   - a) Prove that $4xyzw \leq x^4 + y^4 + z^4 + w^4$. Hint: Use the inequality $2ab \leq a^2 + b^2$ repeatedly.
   - b) Prove that $3abc \leq a^3 + b^3 + c^3$, where a, b, and c are non-negative. Hint: In the inequality of part a) let w be the cube root of xyz.

7. [p. 23 #35] Determine the set of ordered pairs (x, y) of real numbers such that $\frac{x}{y} + \frac{y}{x} \geq 2$.

8. [p. 24 #45] Determine if the rules below define functions from $\mathbb{R}$ to $\mathbb{R}$.
   - a) $f(x) = |x-1|$ if $x < 4$ and $f(x) = |x| - 1$ if $x > 2$.
   - b) $f(x) = |x-1|$ if $x < 2$ and $f(x) = |x| - 1$ if $x > -1$.
   - c) $f(x) = ((x+3)^2 - 9)/x$ if $x \neq 0$ and $f(x) = 6$ if $x = 0$.
   - d) $f(x) = ((x+3)^2 - 9)/x$ if $x > 0$ and $f(x) = x + 6$ if $x < 7$.
   - e) $f(x) = \sqrt{x^2}$ if $x \geq 2$, $f(x) = x$ if $0 \leq x \leq 4$, and $f(x) = -x$ if $x < 0$.

9. [p. 24 #46] Determine the *images* of the functions $f : \mathbb{R} \to \mathbb{R}$ defined below.

   a) $f(x) = \frac{x^2}{1+x^2}$.

   b) $f(x) = \frac{x}{1+|x|}$

10. [p. 24 #55] Let **F** be a field consisting of exactly three distinct elements 0, 1, and x. Prove that $x + x = 1$ and that $x \cdot x = 1$. Obtain the complete addition and multiplication table for **F**.

[Last revised: September 7, 2013]