$$\lim_{\lambda\to\infty}\int_0^\pi f(x)\sin(\lambda x)\,dx = 0$$

We assume that $f(x)$ is continuous on the interval $0 \le x \le \pi$. The function $\sin\lambda x$ is periodic with period $P := 2\pi/\lambda$. Partition the interval $[0,\,\pi]$ into sub-intervals of length P – but an integer number of them will usually not fit exactly. There will usually be a bit left over. Let N be the largest integer so that $NP \le \pi$. Then $NP \le \pi < (N+1)P$.
Let $x_0 = 0$, $x_1 = P$, $x_2 = 2P$, $\ldots$, $x_N = NP$ and rewrite the integral as

$$\begin{aligned} J := \int_0^\pi f(x)\sin(\lambda x)\,dx &= \left(\int_0^{x_1} + \int_{x_1}^{x_2} + \cdots + \int_{x_{N-1}}^{x_N} + \int_{x_N}^{\pi}\right) f(x)\sin(\lambda x)\,dx \\ &= J_1 + J_2 + \cdots + J_N + E. \end{aligned} \tag{1}$$

Except for the last one, each of these integrals is over one full period of $\sin(\lambda x)$. If in the k^{th} interval the function $f(x)$ is a constant, $f(x) = c_k$, then

$$\int_{x_{k-1}}^{x_k} f(x)\sin(\lambda x)\,dx = c_k\int_{x_{k-1}}^{x_k}\sin(\lambda x)\,dx = 0. \tag{2}$$

The idea is that since f is continuous on the closed and bounded interval, it is uniformly continuous there so if P is small (that is, λ is large), f is *almost* a constant in the k^{th} interval – and the integral should be almost zero.

In greater detail, given any $\epsilon > 0$, there is a $\delta > 0$ so that if $|x-y| < \delta$ then $|f(x)-f(y)| < \epsilon$. Now pick λ so large that the length of each interval, P, is less than δ, that is, $2\pi/\lambda < \delta$. Then in the k^{th} interval $f(x) = f(x_k) + [f(x) - f(x_k)]$ where $|f(x) - f(x_k)| < \epsilon$. Therefore

$$\begin{aligned} J_k &:= \int_{x_{k-1}}^{x_k} f(x)\sin(\lambda x)\,dx \\ &= f(x_k)\int_{x_{k-1}}^{x_k}\sin(\lambda x)\,dx + \int_{x_{k-1}}^{x_k}[f(x) - f(x_k)]\sin(\lambda x)\,dx \\ &= J_k' + J_k''. \end{aligned} \tag{3}$$

By equation (2), $J_k' = 0$ while $|J_k''| < \epsilon(x_k - x_{k-1})$.
Also, the last integral, E, in equation (1) is easily estimated as follows. Let $M = \max_{[0,\,\pi]}|f(x)|$. Then, if needed, choosing λ even larger

$$|E| = \left|\int_{x_N}^{\pi} f(x)\sin(\lambda x)\,dx\right| \le M(\pi - x_N) \le MP = 2M\pi/\lambda < \epsilon.$$

Therefore

$$|J| \le |J_1| + |J_2| + \cdots + |J_N| + |E| < \epsilon(x_N - x_0) + \epsilon \le (\pi + 1)\epsilon.$$

Generalization If $f \in C([0, 1])$ and $\varphi\colon \mathbb{R} \to \mathbb{R}$ is continuous and periodic with period P, then

$$\lim_{\lambda\to\infty} \int_0^1 f(x)\varphi(\lambda x)\,dx = \overline{\varphi} \int_0^1 f(x)\,dx.$$

Here $\overline{\varphi} := \frac{1}{P}\int_0^P \varphi(t)\,dt$ is the average of φ over one period.

To prove this, one first does the special case where $\overline{\varphi} = 0$. It is almost identical to what we did above. For the general case let $\psi(t) = \varphi(t) - \overline{\varphi}$. Then $\overline{\psi} = 0$ so we can use the special case and are done.

EXAMPLE Using $\varphi(t) = |\sin t|$ and $f(x) = 1$, this generalization shows how to compute

$$\lim_{\lambda\to\infty} \int_0^1 |\sin(\lambda x)|\,dx.$$

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