

Problem Set 10

DUE: In class Tuesday, Nov. 19. *Late papers will be accepted until 1:00 PM Wednesday.*

REMARK: Please re-read Chapter 16, pages 324-330 on Series of Functions and Chapter 17, pages 337-349 on Integration.

PROBLEMS

1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function. If $f'(x) \geq 0$ everywhere, prove that f is “increasing” in the sense that if $x < y$ then $f(x) \leq f(y)$.
2. [#16.34] Let f be differentiable on the interval $[a, b]$, and suppose that both f and f' are positive. Prove that the function $g := f/(1 + f)$ is bounded and increasing.
3. [#16.49] Suppose that f and g are convex (not necessarily differentiable) and $c \in \mathbb{R}$. Which of the three functions $f + g$, $c \cdot f$, and $f \cdot g$ must be convex? (Give proofs or counterexamples.)
4. Let $f_n(x) := x^n(1 - x)$. Prove that $f_n \rightarrow 0$ uniformly on $[0, 1]$.
5. a) [#16.52] Which polynomials of odd degree are convex on \mathbb{R} ?
b) [#16.52] Characterize the polynomials of degree four that are convex on \mathbb{R} by giving a necessary and sufficient condition on the coefficients.
6. [#16.61] Let $f_n(x) := x^2/(x^2 + n^2)$.
a) Prove that f_n converges pointwise to 0 on \mathbb{R} .
b) Prove that f_n does not converge uniformly to 0 on \mathbb{R} .
c) Does f_n converge uniformly to 0 on the interval $[0, 100]$? Justify your assertion.
7. Use the definition of the integral as a Riemann sum to compute $\int_0^b x^3 dx$. You may use that $\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2}\right)^2$.
8. Use the definition of the integral as a Riemann sum to compute $\int_0^b \cos x dx$. You will need the formula for $\cos \theta + \cos 2\theta + \cos 3\theta + \cdots + \cos n\theta$.
9. [#17.15] Let f be continuous on the interval $[a, b]$ and assume that $f(x) \geq 0$ for all $a \leq x \leq b$. Use the definition of the integral as a Riemann sum to show that if $\int_a^b f(x) dx = 0$, then $f(x) = 0$ everywhere. [You will need to use that since f is continuous, if it is positive at some point, then it is positive in some interval containing the point.]