Math 202 **Exam 3** Jerry L. Kazdan

December 10, 2013 12:00 — 1:20

Directions: Part A has 6 shorter problems (8 points each) while Part B has 4 traditional problems (13 points each). 100 points total].
To receive full credit your solution should be clear and correct. Neatness counts. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3 ×5 with notes on both sides.

Part A: Six shorter problems, 8 points each [total: 48 points]

A-1. Say a function $f(x)$ has the properties $f'(x) = 2\cos 2x$ for all $x \in \mathbb{R}$ and $f(0) = 0$. Show that $f(x) = \sin 2x$. [Hint: To show that "$A = B$", it is often easiest to show that "$A - B = 0$".]

Solution:

Method 1. Let $g(x) := f(x) - \sin 2x$. Then $g'(x) = f'(x) - 2\cos 2x = 0$. Thus by the Mean Value Theorem $g(x) \equiv$constant. But $g(0) = f(0) - 0 = 0$ so $g(x) \equiv 0$.

Method 2. By the Fundamental Theorem of Calculus

$$f(x) = f(0) + \int_0^x f'(t)\,dt = 0 + \int_0^x 2\cos 2t\,dt = \sin 2x.$$

A-2. Find the continuous function f and constant C so that $\displaystyle\int_1^x f(t)\,dt = x\cos(\pi x) + C$.

Solution: Let $x = 0$ to find $0 = 1 \cdot \cos\pi + C = -1 + C$ so $C = 1$.

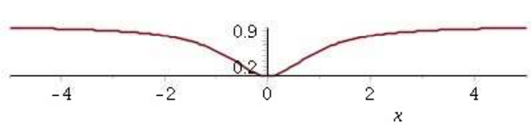
Take the derivative of both sides. By the Fundamental Theorem of Calculus

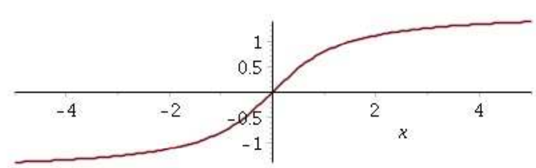
$$f(x) = \cos(\pi x) - \pi x \sin(\pi x).$$

Note that Method 1 is more fundamental since to prove this version of the Fundamental Theorem of Calculus you use exactly the approach of Method 1.

A-3. Give an example of a bounded continuous function $f(x)$, $x \in \mathbb{R}$, that does not attain its supremum. A clear sketch is adequate.

Examples: $\quad y = \dfrac{x^2}{1+x^2} \quad$ and $\quad y = \arctan x$





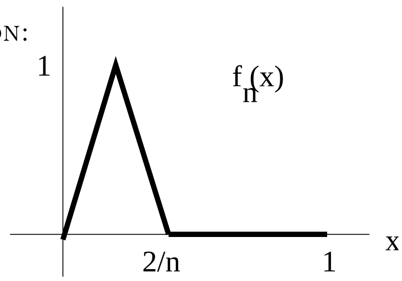
As $x \to +\infty$: in the left example, $y \nearrow 1$, while in the right example $y \nearrow \pi/2$.

A-4. Let a_n be a sequence of real numbers that converges to A. If $a_n \geq 0$, give a clear proof that $A \geq 0$.

SOLUTION: Given any $\varepsilon > 0$, pick n so large that $|a_n - A| < \varepsilon$. Therefore $a_n - A < \varepsilon$. That is, $a_n - \varepsilon < A$. But $a_n \geq 0$. Thus $-\varepsilon < A$. Since ε can be chosen as small as you wish, the only possibility is $0 \leq A$.

A-5. Give an example of a sequence, $f_n(x)$, of functions on the interval [0, 1] that converge pointwise to 0 but do *not* converge uniformly. A good sketch is adequate.

SOLUTION:



If you prefer formulas, another continuous example is $f_n(x) = n^2 x^n (1-x)$ for $0 \leq x \leq 1$, but this is more complicated to see. The above sketch is simpler.

A discontinuous example is to let $f_n(x) := x^n$ for $0 \leq x < 1$ but $f_n(1) = 0$. Note for each $n \geq 0$, that $\sup_{x\in[0,1]} f_n(x) = 1$.

A-6. Let $p(x) = x^9 + a_8 x^8 + \cdots + a_1 x + a_0$. Prove (clearly) that $\lim_{x\to-\infty} p(x) = -\infty$.

SOLUTION: For $x \neq 0$

$$p(x) = x^9 \left[1 + \frac{a_8}{x} + \frac{a_7}{x^2} + \cdots \frac{a_1}{x^8} + \frac{a_0}{x^9}\right].$$

In the limit as $x \to -\infty$, the term in brackets $[1 + \frac{a_8}{x} + \cdots]$ converges to 1 while $x^9 \to -\infty$.

PART B: Four traditional problems, 13 points each [52 points[

B-1. Let $f(x)$ and $g(x)$ be differentiable for $x \in [a, b]$ and let $p \in (a, b)$. Show directly from the definition of the derivative that the product, $f(x)g(x)$, is differentiable at the point p and the derivative is given by the usual rule: $(fg)'(p) = f'(p)g(p) + f(p)g'(p)$.

SOLUTION:

$$\begin{aligned}\frac{f(x+h)g(x+h) - f(x)g(x)}{h} &= \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\ &= \left[\frac{f(x+h) - f(x)}{h}\right] g(x+h) + f(x) \left[\frac{g(x+h) - g(x)}{h}\right]\end{aligned}$$

Now let $h \to 0$.

B-2. Let f be a continuous function on the interval $[a, b]$. If $\displaystyle\int_a^b f(x)\,dx = 0$, show there is a point $c \in (a, b)$ so that $f(c) = 0$.

SOLUTION:

Method 1. By contradiction, if $f(x) \ne 0$ for any $x \in (a, b)$ then because f is continuous, by the Intermediate Value Theorem either $f(x) > 0$ for *all* $x \in (a, b)$ or $f(x) < 0$ for *all* $x \in (a, b)$. But then either $\int_0^1 f(x)\,dx > 0$ or $\int_0^1 f(x)\,dx < 0$, a contradiction.

Same idea but alternate wording: Let $m = \inf_{x\in[a,\,b]} f(x)$ and $M = \sup_{x\in[a,\,b]} f(x)$. Then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a), \qquad \text{so} \qquad m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M. \tag{1}$$

Because f is continuous on $[a, b]$ there are points x_m and x_M in $[a, b]$ where $f(x_m) = m$ and $f(x_M) = M$. By the intermediate value theorem, there are points where f takes oh all values between m and M. Thus, by equation (1) there is a point $c \in [a, b]$ where $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx = 0$.

Method 2. For $t \in [a, b]$, let $g(t) = \displaystyle\int_a^t f(x)\,dx$. Then $g(a) = g(b) = 0$. Also, by the Fundamental Theorem of Calculus, $g(t)$ is differentiable for $t \in (a, b)$ and $g'(t) = f(t)$ (this uses that f is continuous). By Rolle's Theorem there is at least one $c \in (a, b)$ where $g'(c) = 0$. But then $f(c) = 0$.

REMARKS: If f is not required to be continuous, a simple counterexample on the interval $[-1, 1]$ is where $f(x) = -1$ for $x \in [-1, 0)$ while $f(x) = 1$ for $x \in [0, 1]$.

Method 1 also proves the following generalization: Let $p(x) \ge 0$ be any integrable function and let $P := \int_a^b p(x)\,dx$. Assume $P \ne 0$. Then there is a point c where $f(c) = \frac{1}{P}\int_a^b f(x)p(x)\,dx$. Intuitively you can think of $d\mu := p(x)\,dx$ as the element of mass and P as the total mass. [If $P = 0$ the problem is trivial].

B-3. Let $I_k = \{x \in \mathbb{R} \mid a_k \le x \le b_k\}$ be closed bounded *nested* intervals, that is, $I_{k+1} \subseteq I_k$.

a) Use the completeness property of the real numbers ("bounded monotone sequences converge") to show that there is at least one point in the intersection, $\cap I_k$.

   SOLUTION: Since the intervals are nested, $a_k \le a_{k+1}$ and $b_{k+1} \le b_k$. Also $a_k \le b_k \le b_1$ and $b_k \ge a_k \ge a_1$. The a_k are therefore a bounded monotone increasing sequence that converges to some A and similarly the b_k are a bounded monotone decreasing sequence that converges to some $B \le A$. Thus $\cap I_k = [A, B]$.

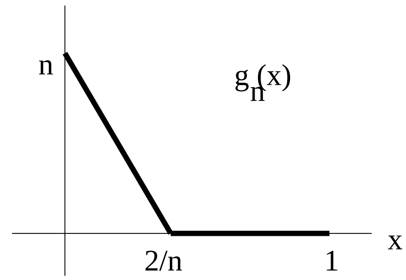
b) Give an example where the intersection is the *whole* interval $-1 \le x \le 1$.

   SOLUTION: Example 1: $I_k = [-1 - \frac{1}{n}, 1 + \frac{1}{n}]$. Example 2: The trivial example $I_1 = I_2 = \cdots = [-1, 1]$.

B-4. Let $f(x)$ be continuous on the interval $[0, 1]$ and $g_n(x)$ be the sequence of functions in the figure. Show that

$$\lim_{n\to\infty} \int_0^1 f(x) g_n(x)\, dx = f(0).$$

SUGGESTION First do the case where $f(x) \equiv 1$.



SOLUTION: Idea: since $\int_0^1 f(x)g_n(x)\,dx = \int_0^{2/n} f(x)g_n(x)\,dx$ the only values of f that matter are those in the small interval $0 \le x \le 2/n$ near $x = 0$. Also, if $h(x) \equiv 1$, then $\int_0^1 h(x)g_n(x)\,dx = \int_0^1 g_n(x)\,dx = 1$. Thus

$$\int_0^1 f(x)g_n(x)\,dx - f(0) = \int_0^{2/n} [f(x) - f(0)]\, g_n(x)\,dx. \tag{2}$$

Because f is continuous, given any $\varepsilon > 0$, there is a $\delta > 0$ so that if $|x - 0| < \delta$, then $|f(x) - f(0)| < \varepsilon$. Pick n so large that $2/n \le \delta$. Then

$$\left| \int_0^{2/n} [f(x) - f(0)]\, g_n(x)\,dx \right| \le \int_0^{2/n} |f(x) - f(0)|\, g_n(x)\,dx < \varepsilon \int_0^{2/n} g_n(x)\,dx = \varepsilon.$$

Using this in equation (2) the conclusion follows.

REMARK: The identical reasoning works for