Math 202 | Exam 2 | Jerry L. Kazdan

November 5, 2013 | 12:00 — 1:20

DIRECTIONS: Part A has 5 shorter problems (8 points each) while Part B has 4 traditional problems (15 points each). [100 points total].
To receive full credit your solution should be clear and correct. Neatness counts. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3 ×5 with notes on both sides.

PART A: Five shorter problems, 8 points each [total: 40 points]

A-1. Give an example of an infinite series $\sum a_n$ that converges but does not converge absolutely. [You do not need to justify your assertion.]

A-2. Give an example of a bounded function defined on $-2 \le x \le 2$ that is continuous everywhere *except* at $x = 0$. [You do not need to justify your assertion].

A-3. Show that the polynomial $p(x) := x^6 + x^5 - 5$ has at least two *real* zeroes.

A-4. Let $g(x)$ be any smooth function and let $f(x) = (x-1)(x-2)(x-3)g(x)$. Show there is a point $c \in (1,\,3)$ where $f''(c) = 0$.

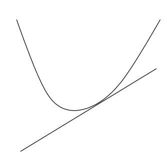
A-5. Say a function $f(x)$ has the properties $f'(x) = 2$ for all $x \in \mathbb{R}$ and $f(1) = 2$. Show that $f(x) = 2x$. [HINT: To show that "$A = B$", it is often easiest to show that "$A - B = 0$".]

PART B: Four traditional problems, 15 points each [60 points[

B-1. Determine if the series $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ converges or diverges. Please explain your reasoning.

B-2. Use the definition of the derivative as the limit of a difference quotient to show that if $f(x) = \cos 2x$, then f is differentiable everywhere and compute its derivative. [You may use that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ and $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0$.]

B-3. Let $f(x)$ have two continuous derivatives in the interval $(a,\,b)$ and say $f''(x) \ge 0$ for all $x \in [a,\,b]$. Prove that for any x_0 the graph of $y = f(x)$ lies above its tangent line at $(x_0,\,f(x_0))$. [If the equation of the tangent line at x_0 is $y = g(x)$, then by "lies above" I mean $f(x) \ge g(x)$ for all $x \in [a,\,b]$.]



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B-4. Suppose a function $f : \mathbb{R} \to \mathbb{R}$ has the property that there is a constant $a > 0$ so that $f'(x) \geq a$ for all $x \in \mathbb{R}$.

a) Show that if $x \geq 0$, then $f(x) \geq f(0) + ax$ while if $x \leq 0$, then $f(x) \leq f(0) + ax$.

b) Show that for every $c \in \mathbb{R}$ there is one (and only one) solution of the equation

$$f(x) = c.$$

Thus, there are two steps: (i) show the equation has at least one solution and (ii) show that the equation has at most one solution.

[NOTE The existence of at least one solution may be *false* if you assume only $f'(x) > 0$. For example the equation $e^x = -1$ has no solution.]