Math 202, Fall 2009 — Jerry L. Kazdan

Homework Set 9 (Due in class on Thursday, Nov. 19)

(late papers accepted until 1:00 Friday)

The problem numbers refer to the D'Angelo-West text.

1. [#16.1] For $x \neq 0$ compute $\displaystyle\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{(x+h)^2}-\frac{1}{x^2}\right)$.

   SOLUTION: Let $f(x)=\frac{1}{x^2}$. Then
$$\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{(x+h)^2}-\frac{1}{x^2}\right)=f'(x)=-\frac{2}{x^3}.$$
   Of course you can also compute the limit directly, but recognizing the limit as a derivative was my thought.

2. [#16.11] Use the definition of the derivative as the limit of a difference quotient to derive the product rule for differentiating $f(x)g(x)$. [SUGGESTION: Add and subtract an appropriate quantity in the numerator.]

   SOLUTION:
$$\begin{aligned}
&\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\
=&\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)}{h}+\frac{f(x+h)g(x)-f(x)g(x)}{h}\\
=&\lim_{h\to 0}f(x+h)\left(\frac{g(x+h)-g(x)}{h}\right)+\left(\frac{f(x+h)-f(x)}{h}\right)g(x)\\
=&\lim_{h\to 0}f(x+h)\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)+\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)\lim_{h\to 0}g(x)\\
=&f(x)g'(x)+f'(x)g(x).
\end{aligned}$$

3. Use the definition of the derivative as the limit of a difference quotient to derive the formula for the derivative of $f(x)=\sqrt{x}$ for $x>0$.

   SOLUTION: For any $x>0$,
$$\begin{aligned}
f'(x)&=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\\
&=\lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)\\
&=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}.
\end{aligned}$$

4. Let a smooth function $g(x)$ have the properties: $g(0) = 3$, $g(1) = 1$, $g(4) = 7$.

   a) Show that at some point $0 < c < 4$ one has $g''(c) > 0$. Better yet, find a number $m > 0$ so that $g''(c) \geq m > 0$.

   b) Is it true that g'' must be positive at at least one point in the interval $0 < x < 1$? Proof or counterexample.

   c) [This is the optimal version of part (a)]. Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be any three points in the plane with $x_1 < x_2 < x_3$, $y_1 > y_2$, and $y_3 > y_2$. Then there is a point $c \in (x_1, x_3)$ such that $g''(c) = m > 0$, where m is the second derivative of the (unique) quadratic polynomial passing through the three points.

   SOLUTION: (a) By the mean value theorem, there is a $c_1 \in (0,1)$ such that $g(1) - g(0) = g'(c_1)(1-0)$ so $g'(c_1) = (g(1) - g(0))/(1-0) = -2$. Similarly, there is a $c_2 \in (1,4)$ such that $g(4) - g(1) = g'(c_2)(4-1)$, so $g'(c_2) = (g(4) - g(1))/(4-1) = 2$.

   Next, by the mean value theorem applied to $g'(x)$ there is $c \in (c_1, c_2)$ such that

$$g''(c) = \frac{(g'(c_2) - g'(c_1)}{c_2 - c_1)} = \frac{4}{c_2 - c_1} > 4/(7-0) = \frac{4}{7}.$$

   (b) False. Consider $g(x) = x^3 - 4x^2 + x + 3$, then $g(0) = 0$, $g(1) = 1$, $g(4) = 7$, but $g''(x) = 6x - 8$ which is negative for $x \in (0,1)$.

   (c) Let $y = p(x) = \alpha x^2 + \beta x + \gamma$ be the quadratic polynomial passing through the three points and let $h(x) = p(x) - g(x)$. Then $h(0) = h(1) = h(4) = 0$. Hence $h'(c_1) = h'(c_2) = 0$ for some $0 < c_1 < 1 < c_2 < 4$. Thus $h''(c) = 0$ for some $c \in (c_1, c_2)$, that is, $g''(c) = p''(c) = 2\alpha > 0$. Note that $2\alpha = p''(c) > 0$ by the same argument used in part (a).

5. Let $v(x)$ be a smooth real-valued function for $0 \leq x \leq 1$. If $v(0) = v(1) = 0$ and $v''(x) \geq 0$ for all $0 \leq x \leq 1$, show that $v(x) \leq 0$ for all $0 \leq x \leq 1$.

   SOLUTION: By contradiction, say $v(x) > 0$ for some $x \in (0,1)$. Then by the mean value theorem $v'(c_1) = \frac{v(x)-v(0)}{x-0} > 0$ for some $c_1 \in (0,x)$ and $v'(c_2) = \frac{v(1)-v(x)}{1-x} < 0$ for some $c_2 \in (x,1)$. Thus, by the Mean Value theorem again, for some $c \in (c_1, c_2)$ we have $v''(c) = \frac{v'(c_2)-v'(c_1)}{c_2-c_1} < 0$. This contradicts our assumption that $v''(x) \geq 0$.

6. Let $g(x)$ is a smooth function with $g(2) = 0$ and let $f(x) = x^2 g(x)$. Use the mean value theorem to show that $f''(c) = 0$ for some $0 < c < 2$.

   SOLUTION: Use the Mean Value Theorem twice. We know $f(0) = 0^2 g(0) = 0$ and $f(2) = 2^2 g(2) = 0$. So $f'(c_1) = 0$ for some $c_1 \in (0,2)$. Since $f'(x) = 2xg(x) + x^2 g'(x)$, $f'(0) = 0 = f'(c_1)$. Hence $f''(c) = 0$ for some $c \in (0, c_1) \subset (0,2)$.

7. a) Let $g(x) := x^3(1-x)$. Use the mean value theorem to show that $g'''(c) = 0$ for some $0 < c < 1$.

b) Let $h(x) := x^3(1-x)^3$. Show that $h'''(x)$ has exactly three distinct roots in the interval $0 < x < 1$.

c) Let $p(x) := \left(\frac{d}{dx}\right)^4 (1-x^2)^4$. Show that p is a polynomial of degree 4 and that it has 4 real distinct zeroes, all lying in the interval $-1 < x < 1$.

SOLUTION: (a) Since $g(0) = g(1) = 0$, there is $c_1 \in (0,1)$ such that $g'(c_1) = 0$. Since $g'(0) = 0 = g'(c_1)$, there is $c_2 \in (0, c_1)$ such that $g''(c_2) = 0$. Since $g''(0) = 0 = g''(c_2)$, there is $c \in (0, c_2)$ such that $g'''(c) = 0$.

(b) Since $h(0) = h(1) = 0$, there is $c_1 \in (0,1)$ such that $h'(c_1) = 0$. Since $h'(0) = h'(c_1) = h'(1) = 0$, there are $c_2 \in (0, c_1)$ and $c_3 \in (c_1, 1)$ such that $h''(c_2) = h''(c_3) = 0$. Since $h''(0) = h''(1) = 0 = h''(c_2) = h''(c_3)$, there are $c_4 \in (0, c_2)$, $c_5 \in (c_2, c_3)$ and $c_6 \in (c_3, 1)$ such that $h'''(c_4) = h'''(c_5) = h'''(c_6) = 0$. So h''' has at least 3 distinct roots $c_4 < c_5 < c_6$. h''' has at most 3 distinct roots because it is a polynomial of degree 3.

(c) Since $h(x) := (1-x^2) - (1+x)^4(1-x)^4$, this problem is almost identical to part (b). Note that $p(x) = h''''(x)$. Since h is a polynomial of degree 8, p is a polynomial of degree 4. Since $h(-1) = h(1) = 0$, there is $c_1 \in (-1,1)$ such that $h'(c_1) = 0$.

Since $h'(-1) = h'(c_1) = h'(1) = 0$, there are $c_2 \in (-1, c_1)$ and $c_3 \in (c_1, 1)$ such that $h''(c_2) = h''(c_3) = 0$.

Since $h''(-1) = h''(1) = 0 = h''(c_2) = h''(c_3)$, there are $c_4 \in (-1, c_2)$, $c_5 \in (c_2, c_3)$ and $c_6 \in (c_3, 1)$ such that $h'''(c_4) = h'''(c_5) = h'''(c_6) = 0$.

Since $h'''(0) = h'''(1) = 0$, there are $c_7 \in (-1, c_4)$, $c_8 \in (c_4, c_5)$, $c_9 \in (c_5, c_6)$, and $c_{10} \in (c_6, 1)$ such that $h''''(c_7) = h''''(c_8) = h''''(c_9) = h''''(c_{10}) = 0$. So $p = h''''$ has at least 4 real distinct roots $c_7 < c_8 < c_9 < c_{10}$. Note that p has at most 4 real distinct roots because it is a polynomial of degree 4.

REMARK: If in part (c) you replace the 4 by n, you get the *Legendre polynomial of degree* n. It has n real distinct zeroes in the interval $(-1, 1)$.

8. If $b \geq 0$, show that for every real c the equation $x^5 + bx + c = 0$ has exactly one real root.

   SOLUTION: Let $f(x) = x^5 + bx + c$. Since $f(a) > 0$ and $f(-a) < 0$ for all sufficiently large $a > 0$, there is $x \in (-a, a)$ such that $f(x) = 0$. Since $b \geq 0$, f is strictly monotone increasing, so f has at most one real root.

9. Let $p(x) := x^3 + 3cx + d$, where c, and d are real. Under what conditions on c and d does this has three distinct real roots? [SUGGESTION: Look at the graph of p and observe something simple about the local maximum and local minimum for p to have three distinct real roots.] [ANSWER: $c < 0$ and $d^2 < -4c^3$].

SOLUTION: Observe that p is strictly monotone increasing when $c \geq 0$ in which case p has exactly one real root. Thus, if p has 3 distinct real roots, then we must have $c \leq 0$, which we now assume. With hindsight it will be simpler if we write $c = -\gamma$ so $\gamma > 0$ and $p(x) = x^3 - 3\gamma x + d$

Since $p'(x) = 3x^2 - 3\gamma$, then p is strictly monotone increasing on $\left(-\infty, -\sqrt{\gamma}\right]$, strictly monotone decreasing on $\left[-\sqrt{\gamma}, \sqrt{\gamma}\right]$, and strictly monotone increasing on $\left[\sqrt{\gamma}, \infty\right)$. Because $p''(x) = 3x$, p has a local maximum at $x = -\sqrt{\gamma}$ and a local minimum at $x = -\sqrt{\gamma}$.

Thus p has at most one root on each of these three intervals. If p has three distinct roots, then p must has exactly one root on each of these intervals. Therefore we need $p\left(-\sqrt{\gamma}\right) > 0$ and $p\left(\sqrt{\gamma}\right) < 0$.

Conversely, by the intermediate value theorem (used thrice), if $p\left(-\sqrt{\gamma}\right) > 0$ and $p\left(\sqrt{\gamma}\right) < 0$ then p has at least (and thus exactly) three distinct roots.

We now compute $p(\pm\sqrt{\gamma})$:

$$p\left(+\sqrt{\gamma}\right) = \gamma\sqrt{\gamma} - 3\gamma\sqrt{\gamma} + d = -2\gamma\sqrt{\gamma} + d.$$

The condition $p(+\sqrt{\gamma}) < 0$ is thus $d < 2\gamma\sqrt{\gamma}$.

Similarly

$$p\left(-\sqrt{\gamma}\right) = 2\gamma\sqrt{\gamma} + d$$

and the condition $p(-\sqrt{\gamma}) > 0$ is $d > -2\gamma\sqrt{\gamma}$.

Combining them we get $-2\gamma\sqrt{\gamma} < d < 2\gamma\sqrt{\gamma}$, that is, $d^2 < 4\gamma^3$.

Summarizing in terms of $c = -\gamma$, $p(x) = x^3 + 3cx + d$ has three real distinct roots if and only if $c < 0$ and $d^2 < -4c^3$.

REMARK. The general cubic polynomial $p(x) := x^3 + Bx^2 + Cx + D$ can be reduced to the special form here by making the substitution $x = t - (B/3)$. You can be led to this by the observation that $p''(x)/3! = (6x + 2B)/6$.

10. [#16.31] Let $f(x)$ be a differentiable function for all real x with the property that $f'(x) < 1$ for all x. Show has at most one *fixed point*, that is, at most one point p where $f(p) = p$.

SOLUTION: Suppose f has two distinct fixed points $a < b$, that is, $f(a) = a$ and $f(b) = b$. Then by the Mean Value Theorem there is $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{b-a}{b-a} = 1$, which contradicts the assumption that $f'(x) < 1$.

11. Let $f(x)$ be a differentiable function for all real x with the property that $|f'(x)| < 1/2$ for all x. Define the sequence x_k by the rule $x_1 = 1$ and $x_{k+1} = f(x_k)$ for $k = 1, 2, \ldots$.

Show that the x_k converge to a point p and that $f(p) = p$, so p is a fixed point of f. [SUGGESTION: Use the mean value theorem to show that

$$|x_{k+1} - x_k| \le \tfrac{1}{2}|x_k - x_{k-1}|$$

and then use work we did earlier to conclude that the x_k is a Cauchy sequence etc.

SOLUTION: For any $k \ge 2$, there is c between x_{k-1} and x_k such that $x_{k+1} - x_k = f(x_k) - f(x_{k-1}) = f'(c)(x_k - x_{k-1})$. Since $|f'(c)| < \frac{1}{2}$, $|x_{k+1} - x_k| = |f'(c)||x_k - x_{k-1}| \le \frac{1}{2}|x_k - x_{k-1}|$. Then x_k converges because of a problem we did before concerning *contracting sequences*.

12. Suppose u is a twice differentiable function on $\mathbb{R}$ which satisfies the differential equation

$$\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} - c(x)u = 0,$$

where $b(x)$ and $c(x)$ are continuous functions on $\mathbb{R}$ with $c(x) > 0$ for every $x \in (0,1)$.

a) Show that u cannot have a positive local maximum in the interval $(0,1)$. Also show that u cannot have a negative local minimum in $(0,1)$.

b) If $u(0) = u(1) = 0$, prove that $u(x) = 0$ for every $x \in [0,1]$.

SOLUTION: (a) Proof by contradiction. Assume that u has a positive local maximum at some point $\alpha \in (0,1)$, then $u''(\alpha) \le 0$, $b(\alpha)u'(\alpha) = 0$, and $-c(\alpha)u(\alpha) < 0$. Adding these three (in)equalities, we get $u''(\alpha) + b(\alpha)u'(-c(\alpha)u(\alpha) < 0$, but u satisfies $u'' + b(x)u' - c(x)u = 0$ so we have a contradiction. Thus u has no positive local maximum on $(0,1)$.

Assume that u has a negative local minimum at $c \in (0,1)$, then the function $v(x) := -u(x)$ also satisfies the same equation, $v'' + bv' - cv = 0$ and would have a positive local maximum – which cannot happen by the previous paragraph. So u has no negative local minimum on $(0,1)$.

(b) Since u is continuous on the closed and bounded interval $[0,1]$, u attains its maximum somewhere on $[0,1]$ at some $c \in [0,1]$. We have $u(c) \ge u(0) = 0$. If $u(c) > 0$, since $u(x) = 0$ at the end points of [0, 1], then u has a positive *local* maximum at $c \in (0,1)$, which is impossible by part (a). Hence $u(c) = 0$, that is $u \le 0$ on $[0,1]$. Similarly, $u \ge 0$ on $[0,1]$. Consequently $u(x) \equiv 0$ on $[0,1]$.

[Last revised: November 2, 2013]