Math 202 Jerry L. Kazdan

Homework Set 9 (Due in class on Thursday, Nov. 19)

(late papers accepted until 1:00 Friday)

The problem numbers refer to the D'Angelo-West text.

1. [#16.1] For $x \neq 0$ compute $\displaystyle\lim_{h\to 0} \frac{1}{h}\left(\frac{1}{(x+h)^2} - \frac{1}{x^2}\right)$.

2. [#16.11] Use the definition of the derivative as the limit of a difference quotient to derive the product rule for differentiating $f(x)g(x)$. [SUGGESTION: Add and subtract an appropriate quantity in the numerator.]

3. Use the definition of the derivative as the limit of a difference quotient to derive the formula for the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

4. Let a smooth function $g(x)$ have the three properties: $g(0) = 3$, $g(1) = 1$, $g(4) = 7$.
   a) Show that at some point $0 < c < 4$ one has $g''(c) > 0$. Better yet, find a number $m > 0$ so that $g''(c) \geq m > 0$.
   b) Is it true that g'' must be positive at at least one point in the interval $0 < x < 1$? Proof or counterexample.
   c) [This is the optimal version of part (a)]. Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be any three points in the plane with $x_1 < x_2 < x_3$, $y_1 > y_2$, and $y_3 > y_2$. Then there is a point $c \in (x_1, x_3)$ such that $g''(c) = m > 0$, where m is the second derivative of the (unique) quadratic polynomial passing through the three points.

5. Let $v(x)$ be a smooth real-valued function for $0 \leq x \leq 1$. If $v(0) = v(1) = 0$ and $v''(x) > 0$ for all $0 \leq x \leq 1$, show that $v(x) \leq 0$ for all $0 \leq x \leq 1$.

6. Let $g(x)$ is a smooth function with $g(2) = 0$ and let $f(x) = x^2 g(x)$. Use the mean value theorem to show that $f''(c) = 0$ for some $0 < c < 2$.

7. a) Let $g(x) := x^3(1-x)$. Use the mean value theorem to show that $g'''(c) = 0$ for some $0 < c < 1$.
   b) Let $h(x) := x^3(1-x)^3$. Show that $h'''(x)$ has exactly three distinct roots in the interval $0 < x < 1$.

c) Let $p(x) := \left(\frac{d}{dx}\right)^4 (1-x^2)^4$. Show that p is a polynomial of degree 4 and that it has 4 real distinct zeroes, all lying in the interval $-1 < x < 1$.

8. If $b \geq 0$, show that for every real c the equation $x^5 + bx + c = 0$ has exactly one real root.

9. Let $p(x) := x^3 + 3cx + d$, where c, and d are real. Under what conditions on c and d does this has three distinct real roots? [SUGGESTION: Look at the graph of p and observe something simple about the local maximum and local minimum for p to have three distinct real roots.] [ANSWER: $c < 0$ and $d^2 < -4c^3$].

10. [#16.31] Let $f(x)$ be a differentiable function for all real x with the property that $f'(x) < 1$ for all x. Show has at most one *fixed point*, that is, at most one point p where $f(p) = p$.

11. Let $f(x)$ be a differentiable function for all real x with the property that $|f'(x)| < 1/2$ for all x. Define the sequence x_k by the rule $x_1 = 1$ and $x_{k+1} = f(x_k)$ for $k = 1, 2, \ldots$. Show that the x_k converge to a point p and that $f(p) = p$, so p is a fixed point of f. [SUGGESTION: Use the mean value theorem to show that

$$|x_{k+1} - x_k| \leq \tfrac{1}{2}|x_k - x_{k-1}|$$

and then use work we did earlier to conclude that the x_k is a Cauchy sequence etc.

12. Suppose u is a twice differentiable function on $\mathbb{R}$ which satisfies the differential equation

$$\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} - c(x)u = 0,$$

where $b(x)$ and $c(x)$ are continuous functions on $\mathbb{R}$ with $c(x) > 0$ for every $x \in (0,1)$.

a) Show that u cannot have a positive local maximum in the interval $(0,1)$. Also show that u cannot have a negative local minimum in $(0,1)$.

b) If $u(0) = u(1) = 0$, prove that $u(x) = 0$ for every $x \in [0,1]$.

[Last revised: November 1, 2013]