Math 202, Fall 2009 Jerry L. Kazdan

Homework Set 8 (Due in class on Thursday, Nov. 12)

(late papers accepted until 1:00 Friday)

The problem numbers refer to the D'Angelo-West text.

For the True-False [T/F] questions, either provide a proof or give a counterexample.

1. [#15.13] Use ϵ-δ to show that the function $|x|$ is continuous for all $x \in \mathbb{R}$.

   SOLUTION: Use three cases, Case 1: At a point $x = a > 0$. Then this is the function x, which we know is continuous – as long as we choose a small enough interval about a in which $x >$. For this choose δ so small that in the interval $|x - a| < \delta$ *both* $x > 0$ *and* $|x - a| < \varepsilon$. The first condition is satisfied if, say, $\delta < a/2$ since then $x > a/2$. The second condition is satisfied if $\delta \le \varepsilon$. If *both* of these are satisfied, then $|x| = x$ so that $||x| - a| = |x - a| < \varepsilon$.

   Case 2: At a point $x = a < 0$. Since $|-x| = |x|$, the functions $|x|$ and $|-x|$ are identical so we can reduce to Case 1.

   Case 3: At the point $x = 0$, pick $\delta = \varepsilon$. Then, if $|x| < \delta$, we also have $|x - a| < \varepsilon$.

2. [#15.2] [T/F] There is a continuous $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ if and only if x is an integer.

   SOLUTION: True. $f(x) = \sin(\pi x)$ or $|\sin \pi x|$ etc. Or just draw a sketch.

3. [#15.3] [T/F] If $f : \mathbb{R} \to \mathbb{R}$ is continuous everywhere and $f(x) = 0$ for all rational numbers x, then $f(x) = 0$ for all real x.

   SOLUTION: True. For any $a \in \mathbb{R}$, there is a sequence of rational numbers $x_n \to a$. Since f is continuous, $f(a) = \lim_{n\to\infty} f(x_n) = 0$.

4. [#15.4] [T/F] There exists $x > 1$ such that $\frac{x^2+5}{3+x^7} = 1$.

   SOLUTION: True. Let $f(x) = \frac{x^2+5}{3+x^7}$. Since $f(1) = \frac{6}{4} > 1$ and since $f(2) = \frac{9}{131} < 1$, there is $x \in (1, 2)$ such that $f(x) = 1$.

   ALTERNATE: This is equivalent to showing that the polynomial $p(x) = x^2 + 5 - 3 - x^7$ has a root st some point $x > 1$. Clearly $p(1) = 2 > 0$ while $p(2) = 2^2 + 5 - 3 - 2^7 < 0$. Now use the intermediate value theorem.

5. [#15.5] [T/F] The function $f(x) := |x|^3$ is continuous for all $x \in \mathbb{R}$.

SOLUTION: True. METHOD 1: Write $|x|^3 = x^2|x|$. This is the product of two continuous functions.

METHOD 2: Let $f(x) = x^3$ and $g(x) = |x|$. f is continuous since it is a polynomial. g is continuous by problem 1. So their composition $g \circ f$ is continuous, that is, $|x^3|$ is continuous.

6. [#15.7] [T/F] Let f, g, and h be continuous on the interval $[0, 2]$. If $f(0) < g(0) < h(0)$ and $f(2) > g(2) > h(2)$, then there exists some $c \in [0, 2]$ such that $f(c) = g(c) = h(c)$.

   SOLUTION: False. The easiest way to see this is to try to draw a sketch. You will immediately find a counterexample to the statement.

   It is then easy to do this using formulas. For instance, let $f(x) = 2x$, $g(x) = 1$ and $h(x) = 4-2x$. Then $f(0) < g(0) < h(0)$ and $f(2) > g(2) > h(2)$. If $f(x) = g(x) = h(x)$ for some x, then $2x = 1 = 4 - 2x$, which is impossible.

7. [#15.8] [T/F] Let $f : \mathbb{R} \to \mathbb{R}$. If $|f|$ is continuous, then f is continuous.

   SOLUTION: False. Define $f(x) = 1$ when $x > 0$ and $f(x) = -1$ when $x \leq 0$, then $|f(x)| = 1$ is continuous but $f(x)$ is not.

8. [#15.10][T/F]

   a) If f is continuous on $\mathbb{R}$, then f is bounded.

   b) If f is continuous on $[0, 1]$, then f is bounded.

   c) If f is continuous on $\mathbb{R}$ and is bounded, then f attains its supremum.

   SOLUTION: (a) False. $f(x) = x$ is continuous on $\mathbb{R}$, but f is unbounded.

   (b) True. Theorem 15.24.

   (c) False. $f(x) = x^2/(1 + x^2)$ is continuous on $\mathbb{R}$, its supremum is $+1$ but does not attain its supremum.

9. [#15.15] Let $f(x) := x^2 + 4x$. Clearly $\lim_{x\to 0} f(x) = 0$. Assuming that $0 < \epsilon < 4$, how small must δ be so that $|x| < \delta$ implies that $|f(x)| < \epsilon$? Express δ as a function of ϵ.

   SOLUTION: Since $f(x) = x(x + 4)$ we put a preliminary restriction on x to control the term $x+4$. For instance, require that $|x| < 1$ so we are requiring that $\delta < 1$. If $|x| < \delta$ then

   $$|f(x)| = |x^2 + 4x| \leq |x||x + 4| < 5\delta.$$

   To insure that $|f(x)| < \varepsilon$ in the interval we also choose $5\delta \leq \varepsilon$. Consequently, choose $\delta = \min(1, \frac{\varepsilon}{5})$.

10. [#15.12] Construct a function f with the property that there are sequences a_n and b_n converging to zero such that $f(a_n)$ converges to zero but $f(b_n)$ is unbounded.

    Does there exist such a function f that is continuous at $x = 0$?

    SOLUTION: Define $f(x) = 0$ for $x \leq 0$ and $f(x) = \frac{1}{x}$ for $x > 0$. Let $a_n = 0$ and $b_n = \frac{1}{n}$. Then $f(a_n) = 0$ converges to zero and $f(b_n) = n$ is unbounded.

    ANOTHER EXAMPLE. Let $f(x) = \frac{1}{x}\sin\frac{1}{x}$ for $x > 0$ Since $\sin\theta = 0$ at $\theta = n\pi$, $n = 1, 2, \ldots$ and $\sin\theta = 1$ at $\theta = \frac{\pi}{2} + 2n\pi$, $n = 1, 2, \ldots$, let $a_n = 1/(n\pi)$ and $b_n = 1/\left(\frac{\pi}{2} + 2n\pi\right)$.

    When f is continuous, there is no such f since $\lim_{n\to\infty} f(b_n) = f(0)$.

11. [#15.17] Let $f(a, n) := (1 + a)^n$, where a and n are positive.

    a) For constant a, how does $f(a, n)$ behave as $n \to \infty$? For constant n, how does $f(a, n)$ behave as $a \to 0$?

    b) Let $L \geq 1$ be a given real number. Prove that there exists a sequence $a_n \to 0$ and $f(a_n, n) \to L$ as $n \to \infty$. In other words, depending on the choice of a_n, f may approach any value.

    SOLUTION: (a) Since $(1 + a) > 1$, then $(1 + a)^n$ diverges to infinityas $n \to \infty$.

    If we fix n, $f(a, n)$ is a polynomial of degree n, and hence continuous. So $\lim_{a\to 0} f(a, n) = f(0, n) = 1$.

    (b) We can pick a_n such that $f(a_n, n) = L$, that is, $(1 + a_n)^n = L$ Solving this we get $a_n = L^{\frac{1}{n}} - 1$ Note that with this choice $a_n \to 0$ as $n \to \infty$.

12. Given any real number $c > 0$, prove there is an $x > 0$ such that $x^{17} = c$.

    SOLUTION: Let $f(x) = x^{17}$, then $f(0) = 0 < c$. $f(1 + c) = (1 + c)^{17} \geq 1 + 17c > c$. Hence there is $x \in (0, 1 + c)$ such that $f(x) = c$.

    This is a special case of the fact that any polynomial $p(x$ whose degree is odd has at least one real zero (show that for large enough x, then $p(\pm x)$ have opposite sign. The result then follows from the intermediate value theorem.

13. [#15.21] Prove that there exists $x \in [1, 2]$ such that $x^5 + 2x + 5 = x^4 + 10$.

    SOLUTION: This is also an easy consequence of the intermediate value theorem. Let $f(x) = x^5 + 2x + 5 - x^4 - 10$. Then $f(1) = -3 < 0$ and $f(2) = 15 > 0$. So there is $x \in (1, 2)$ such that $f(x) = 0$. In other words, $x^5 + 2x + 5 = x^4 + 10$ holds for this x.

[Last revised: October 30, 2013]