Homework Set 8 (Due in class on Thursday, Nov. 12)

(late papers accepted until 1:00 Friday)

The problem numbers refer to the D'Angelo-West text.

For the True-False [T/F] questions, either provide a proof or give a counterexample.

1. [#15.13] Use ϵ-δ to show that the function $|x|$ is continuous for all $x \in \mathbb{R}$.

2. [#15.2] [T/F] There is a continuous $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ if and only if x is an integer.

3. [#15.3] [T/F] If $f : \mathbb{R} \to \mathbb{R}$ is continuous everywhere and $f(x) = 0$ for all rational numbers x, then $f(x) = 0$ for all real x.

4. [#15.4] [T/F] There exists $x > 1$ such that $\frac{x^2+5}{3+x^7} = 1$.

5. [#15.5] [T/F] The function $f(x) := |x|^3$ is continuous for all $x \in \mathbb{R}$.

6. [#15.7] [T/F] Let f, g, and h be continuous on the interval $[0,\, 2]$. If $f(0) < g(0) < h(0)$ and $f(2) > g(2) > h(2)$, then there exists some $c \in [0,\, 2]$ such that $f(c) = g(c) = h(c)$.

7. [#15.8] [T/F] Let $f : \mathbb{R} \to \mathbb{R}$. If $|f|$ is continuous, then f is continuous.

8. [#15.10][T/F]

   a) If f is continuous on $\mathbb{R}$, then f is bounded.

   b) If f is continuous on $[0, 1]$, then f is bounded.

   c) If f is continuous on $\mathbb{R}$ and is bounded, then f attains its supremum.

9. [#15.15] Let $f(x) := x^2 + 4x$. Clearly $\lim_{x\to 0} f(x) = 0$. Assuming that $0 < \epsilon < 4$, how small must δ be so that $|x| < \delta$ implies that $|f(x)| < \epsilon$? Express δ as a function of ϵ.

10. [#15.12] Construct a function f with the property that there are sequences a_n and b_n converging to zero such that $f(a_n)$ converges to zero but $f(b_n)$ is unbounded.

    Does there exist such a function f that is continuous at $x = 0$?

11. [#15.17] Let $f(a,n) := (1+a)^n$, where a and n are positive.

   a) For constant a, how does $f(a,n)$ behave as $n \to \infty$? For constant n, how does $f(a,n)$ behave as $a \to 0$?

   b) Let $L \geq 1$ be a given real number. Prove that there exists a sequence $a_n \to 0$ and $f(a_n, n) \to L$ as $n \to \infty$. In other words, depending on the choice of a_n, f may approach any value.

12. Given any real number $c > 0$, prove there is an $x > 0$ such that $x^{17} = c$.

13. [#15.21] Prove that there exists $x \in [1,2]$ such that $x^5 + 2x + 5 = x^4 + 10$.

[Last revised: November 13, 2009]