Math 202 Jerry L. Kazdan

Homework Set 7 (Due in class on Thursday, Nov. 5)

(late papers accepted until 1:00 Friday)

The problem numbers refer to the D'Angelo-West text.

1. [#14.50] Determine if $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\cdots$ converges.

   SOLUTION: Since $\frac{1}{2n-1}>\frac{1}{2}\cdot\frac{1}{n}$ and since the harmonic sequence $\sum\frac{1}{n}$ diverges, $\sum\frac{1}{2n-1}$ diverges by the comparison test.

2. In class we proved that the harmonic series $\sum\frac{1}{n}$ diverges. Use the estimate in our proof to find an integer N so that

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{N}>100.$$

   SOLUTION: Let $S_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$. Group the terms as we did in class (and in the book):

$$\begin{aligned} S_{32} =&1+\frac{1}{2}+\left[\frac{1}{3}+\frac{1}{4}\right]+\left[\frac{1}{5}+\cdots+\frac{1}{8}\right]+\left[\frac{1}{9}+\cdots+\frac{1}{16}\right]+\left[\frac{1}{17}+\cdots+\frac{1}{32}\right]\\ >&1+\frac{1}{2}\quad+\quad\frac{1}{2}\quad+\quad\frac{1}{2}\quad+\quad\frac{1}{2}\quad+\quad\frac{1}{2}\;=\;1+\frac{5}{2}\end{aligned}.$$

   To see the pattern, rewrite this as

$$\begin{aligned} S_{32} =&1+\frac{1}{2}+\left[\frac{1}{3}+\frac{1}{2^2}\right]+\left[\frac{1}{5}+\cdots+\frac{1}{2^3}\right]+\left[\frac{1}{9}+\cdots+\frac{1}{2^4}\right]+\left[\frac{1}{17}+\cdots+\frac{1}{2^5}\right]\\ >&1+5\frac{1}{2}\end{aligned}.$$

   Similarly, $S_2\ge 1+1/2$, $S_4>1+2/2$, $S_8>1+3/2$, and $S_{16}>1+4/2$. It should be fairly clear that $S_{2^n}\ge 1+n/2=(n+2)/2$. We prove this formally by induction.

   This is true for $n=0$ because $S_{2^0}=S_1=1=\frac{0+2}{2}$. If $S_{2^k}\ge\frac{k+2}{2}$ for some $k\ge 0$, then

$$\begin{aligned} S_{2^{k+1}} =&S_{2^k}+\sum_{\ell=2^k+1}^{2^{k+1}}\frac{1}{\ell}\ge\frac{k+2}{2}+\sum_{\ell=2^k+1}^{2^{k+1}}\frac{1}{\ell}\\ >&\frac{k+2}{2}+\frac{1}{2}=\frac{k+3}{2}.\end{aligned}$$

   Hence $S_{2^n}\ge\frac{n+2}{2}$ for any $n\ge 0$. In particular, $S_{2^{198}}\ge\frac{198+2}{2}=100$.

3. [#14.59] Let a_n be a convergent sequence of positive real numbers. Prove that $\sum_1^\infty \frac{1}{na_n}$ diverges.

SOLUTION: Since the sequence a_n converges, it is bounded above by some number $M > 0$. Then $na_n \leq nM$ so $\sum \frac{1}{na_n} \geq M \sum \frac{1}{n}$ which diverges.

4. [#14.60] Determine if each of the series below converges or diverges.

$$(a) \quad \sum_1^\infty \frac{2n^2+15n+2}{n^4+3n+1} \qquad (b) \quad \sum_1^\infty \frac{2n^2+15n+2}{n^3+3n+1} \qquad (c) \quad \sum_1^\infty \frac{3+5n+n^2}{2^n}$$

SOLUTION: Since $\lim \frac{2n^2+15n+2}{n^4+3n+1} \Big/ \frac{1}{n^2} = 2$ and since $\sum \frac{1}{n^2}$ converges, then (a) converges by the comparison test.

Since $\lim \frac{2n^2+15n+2}{n^3+3n+1} \Big/ \frac{1}{n} = 2$ and since $\sum \frac{1}{n}$ diverges, then (b) diverges by the comparison test.

Since $\lim \frac{3+5n+n^2}{2^n} \Big/ \frac{1}{n^2} = 0$ and since $\sum \frac{1}{n^2}$ converges, (c) converges by the comparison test. Note that the ratio test also works for (c).

5. Find the disk of convergence (in the complex plane) of the following power series:

$$(a) \quad \sum_0^\infty \frac{n^2(z-2)^{2n}}{3^n} \qquad (b) \quad \sum_0^\infty \frac{n^2(z-2i)^n}{3n!} \qquad (c) \quad \sum_0^\infty n!(z+3)^n$$

Note: only find the center and radius of this disk. You are not being asked to determine the convergence on the boundary of the disk of convergence.

SOLUTION: (a) The ratio is

$$\left| \frac{(n+1)^2(z-2)^{2n+1}}{3^{n+1}} \Big/ \frac{n^2(z-2)^{2n}}{3^n} \right| = \frac{(n+1)^2}{n^2} \frac{|z-2|}{3} \to \frac{|z-2|}{3} \tag{1}$$

as $n \to \infty$. So the series converges when $\frac{|z-2|}{3} < 1$, that is, $|z-2| < 3$. Hence the center of the disk of convergence is $z = 2$ and its radius is 3

(b) The ratio is

$$\left| \frac{(n+1)^2(z-2i)^{n+1}}{3(n+1)!} \Big/ \frac{n^2(z-2i)^n}{3n!} \right| = \frac{(n+1)^2|z-2i|}{n^2(n+1)(n+2)(n+3)} \to 0 \tag{2}$$

as $n \to \infty$. Hence the disk of convergence is the whole plane.

(c) When $z = -3$, the series obviously converges. When $z \neq -3$, the ratio is $n|z+3| > 1$ when n is large, so the series diverges.

6. [#14.52] ALTERNATING SERIES TEST Let c_k be a sequence of positive real numbers that is monotone decreasing to zero. Show that the alternating series

$$c_1 - c_2 + c_3 - c_4 + c_5 - c_6 + \cdots$$

converges. [As a model, look at the proof we did in class that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ converges.]

SOLUTION: Call the partial sum $S_n = c_1 - c_2 + \cdots \pm c_n$. Since $S_{2(n+1)} - S_{2n} = c_{2n+1} - c_{2n} \geq 0$, the sequence of even terms, S_{2n}, is monotone increasing. Similarly, $S_{2(n+1)-1} - S_{2n-1} = -c_{2n} + c_{2n+1} \leq 0$, so the sequence of odd terms, S_{2n+1}, is monotone decreasing. Also $S_{2n+1} - S_{2n} = c_{2n+1} > 0$. This implies that

$$S_2 < S_4 < \cdots < S_{2n} < S_{2n+1} < \cdots < S_1. \tag{3}$$

Thus, both the odd and even sequences are monotone and bounded so they converge. Say $S_{2n} \nearrow \alpha$ and $S_{2n+1} \searrow \beta$. But $\beta - \alpha < S_{2n+1} - S_{2n} = c_{2n+1} \to 0$ as $n \to \infty$. Consequently, $\alpha = \beta$ and the S_n converges to α. Moreover, the inequality (3) showa that the error, $|S_n - \alpha|$ is ai most the first neglected term: $|S_n - \alpha| < c_{n+1}$.

7. [e IS IRRATIONAL]. We defined the number e by the power series

$$e = \sum_0^\infty \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} + \cdots \tag{4}$$

The point of this problem is to show that e is irrational.

a) Show that $2 < e < 3$. So e is definitely not an integer.

b) By contradiction, say $e = \frac{p}{q}$, where p and q are positive integers with $q \geq 2$. Show that

$$e\, q! = N + \frac{c}{q+1}, \tag{5}$$

where N is an integer and $0 < c < e$. Thus, conclude that $\frac{c}{q+1}$ must be an integer.

c) Then show that this contradicts $e < 3$ and $q + 1 \geq 3$.

SOLUTION: (a) It is obvious that $e > 2$. To show that $e < 3$ we compare with a geometric series

$$\begin{aligned} e &= 1 + \left(1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} + \cdots\right) \\ &< 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots\right) = 1 + \frac{1}{1 - \frac{1}{2}} = 3. \end{aligned} \tag{6}$$

(b) By contradiction, assume e is rational, say $e = \frac{p}{q}$ for some integers p and q. Because e is not an integer, $q \geq 2$. Multiply both sides of (4) by $q!$ and note that

$eq! = p(q-1)!$ is an integer. Now split the infinite series into two parts to find

$$\begin{aligned} e\,q! &= q!\left[1+1+\frac{1}{2!}+\cdots+\frac{1}{q!}\right]+q!\left[\frac{1}{(q+1)!}+\frac{1}{(q+2)!}+\frac{1}{(q+3)!}+\frac{1}{(q+4)!}+\cdots\right] \\ &= N+\frac{1}{q+1}\left[1+\frac{1}{q+2}+\frac{1}{(q+2)(q+3)}+\frac{1}{(q+2)(q+3)(q+4)}+\cdots\right] \end{aligned}$$

In the first line we have observed that the first group of terms is some integer, N.

Let

$$c := 1+\frac{1}{q+2}+\frac{1}{(q+2)(q+3)}+\frac{1}{(q+2)(q+3)(q+4)}+\cdots. \tag{7}$$

To complete part (b) we need to show that $1 < c < e$. But, except for the additional first term, 1, equation (4) is the special case of (7) with $q = 0$. Thus $c < e < 3$ (in fact we have shown that $c < 2$). [The estimate (6) also shows that $c < 2$.]

(c) From equation (5), since we assumed that $e = p/q$, then $\frac{c}{(q+1)}$ is an integer. But, as noted above, $q \geq 2$. Thus $\frac{c}{q+1} < 1$, so it cannot be an integer. This is a contradiction.

[Last revised: October 26, 2013]