Math 202, Fall 2009 Jerry L. Kazdan

Homework Set 6 (Due in class on Thursday, Oct. 29)

(late papers accepted until 1:00 Friday)

The problem numbers refer to the D'Angelo-West text.

1. [#14.24] Let $f(x) := x^2 - 4x + 6$ and let x_n be a sequence defined by the recurrence $x_{n+1} = f(x_n)$, that is, $x_{n+1} = x_n^2 - 4x_n + 6$.

   a) If $\lim_{n\to\infty}$ exists and equals L, what possible values can L have?

   b) The behavior as $n \to \infty$ depends on the initial value, x_0. For each $x_0 \in \mathbb{R}$, describe this behavior. [Hint: Graph the functions $y = x$ and $y = f(x)$ and interpret the graphs. In this example, it may help to rewrite $y = f(x)$ as $y - 2 = (x - 2)^2$.]

   Solution: (a) Suppose the limit exist and let us call it A. As $n \to \infty$, $x_{n+1} = x_n^2 - 4x_n + 6$ becomes $A = A^2 - 4A + 6$. So $A = 2$ or $A = 3$.

   (b) When $x_0 > 3$, x_n will be monotone increasing, so x_n does not have a chance to converge to 2 or 3. When $x_0 < 1$, $x_1 > 3$ so the same story goes. When $x_0 = 3$ or $x_0 = 1$, $x_n = 3$ for any $n > 0$. When $x_0 \in [2, 3)$, x_n is monotone decreasing and bounded below by 2, so x_n converges to 2. When $x_0 \in (1, 2]$, $x_1 \in [2, 3)$, so x_n converges to 2.

2. [#14.27] For $c > 0$, let $x_n = (c^n + 1)^{1/n}$. Determine $\lim_{n\to\infty} x_n$. More generally, find $\lim_{n\to\infty}(a^n + b^n)^{1/n}$. [Hint: First consider the case $c < 1$ and use the Squeeze Theorem.]

   Solution: When $c \le 1$, since $1 \le x_n \le 2^{\frac{1}{n}}$,then $\lim_{n\to\infty} x_n = 1$ by the squeeze theorem. When $c > 1$, $\lim_{n\to\infty}(c^n + 1)^{1/n} = c \lim_{n\to\infty}(1 + (\frac{1}{c})^n)^{1/n} = c$.

   If $a \ge b \ge 0$, then $\lim_{n\to\infty}(a^n + b^n)^{1/n} = a \lim_{n\to\infty}(1 + (\frac{b}{a})^n)^{1/n} = a$. Similarly, if $b \ge a \ge 0$, So $\lim_{n\to\infty}(a^n + b^n)^{1/n} = b$. If $a \ge 0 \ge b$ and $|b| > |a|$, then $(a^n + b^n)^{1/n}$ is not well-defined. If $a \ge 0 \ge b$ and $|b| = |a|$, then $\lim_{n\to\infty}(a^n + b^n)^{1/n}$ does not exist. If $a \ge 0 \ge b$ and $|b| < |a|$, then $\lim_{n\to\infty}(a^n + b^n)^{1/n} = a$ because $a(1 - \left(\frac{|b|}{|a|}\right)^n) \le a(1 + (\frac{b}{a})^n)^{\frac{1}{n}} = (a^n + b^n)^{1/n} < 2^{1/n}a$.

3. [#14.30] Let x_n be the sequence defined recursively by $x_1 = 1$ and $x_{n+1} = 1/(x_1 + \cdots + x_n)$. Prove that this sequence converges and obtain the limit.

   Solution: Clearly x_n is monotone decreasing and bounded below by 0, so the limit exist.

Call the limit A, then $A \geq 0$ and $x_n \geq A$ for all $n \in \mathbb{N}$. Therefore $x_1 + \cdots + x_n > nA$. If $A > 0$, then $x_{n+1} \leq 1/(nA) \to 0$ as $n \to \infty$, which implies that $A = 0$. So the limit is 0.

Slight alternate approach. Since $x_1 = 1$ and the sequence is decreasing, $x_1 + x_2 + \cdots + x_n < n$. Therefore $x_{n+1} > 1/n$. Consequently, by comparison with the harmonic series, the series $\sum_1^n \frac{1}{k}$ the series $x_1 + x_2 + \cdots$ diverges to $+\infty$ so the $x_{n+1} = 1/\left(\sum_1^n x_k\right) \to 0$.

4. [#14.33] If $\displaystyle\sum_{k=1}^{\infty} a_k$ converges to A and $\displaystyle\sum_{k=1}^{\infty} b_k$ converges to B, show that $\displaystyle\sum_{k=1}^{\infty}(a_k + b_k)$ converges to $A + B$.

Solution: For any $\varepsilon > 0$, there is $N_1, N_2 \in \mathbb{N}$ such that $\left|\displaystyle\sum_{k=1}^{n} a_k - A\right| < \frac{1}{2}\varepsilon$ when $n \geq N_1$ and that $\left|\displaystyle\sum_{k=1}^{n} b_k - B\right| < \frac{1}{2}\varepsilon$ when $n \geq N_2$. Let $N = \max(N_1, N_2)$, when $n \geq N$, $\left|\displaystyle\sum_{k=1}^{n}(a_k + b_k) - (A + B)\right| \leq \left|\displaystyle\sum_{k=1}^{n} a_k - A\right| + \left|\displaystyle\sum_{k=1}^{n} b_k - B\right| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$.

5. [#14.36] Find the rational number whose repeating decimal expansion is $.247247247\ldots$. Also, find the rational number whose repeating octal (that is, base 8) expansion is $.247247247\ldots$.

Solution: Write q as

$$q = \frac{247}{10^3} + \frac{247}{10^6} + \frac{247}{10^9} + \cdots.$$

This is a geometric series of the form $a(1 + c + c^2 + c^3 + \cdots)$ where $a = 247/1000$ and $c = 1/1000 < 1$. The sum is

$$q = a/(1-c) = \frac{247}{10^3}\left(\frac{1}{1 - 10^{-3}}\right) = \frac{247}{10^3 - 1} = \frac{247}{999}.$$

Similarly, if $q = .247247\ldots$ (octal expansion), then $q = a(1 + c + c^2 + \cdots)$ with $a = 247/8^3$ and $c = 1/8^3$. Note that $8^3 = 1000$ octal and $8^3 - 1 = 777$ octal.

$$q = a/(1-c) = \frac{247}{8^3}\left(\frac{1}{1 - 8^{-3}}\right) = \frac{247}{8^3 - 1} = \frac{247}{777}.$$

6. [#14.43] Compute $\sum_{k=1}^{\infty}\left(\frac{x}{x+1}\right)^k$. What assumptions must be made about x?

Solution: By ratio test, this series converges if $|\frac{x}{x+1}| < 1$, and we know from a previous problem set that this means $x > -\frac{1}{2}$. Note that when $x = \frac{1}{2}$ the series becomes $\sum(-1)^n$, which obviously does not converges since $(-1)^n$ does not converge to 0.

When it converges, this is just the geometric series and the limit is

$$\frac{\frac{x}{x+1}}{1-\frac{x}{x+1}} = x.$$

7. [Telescoping Series #14.44]

   a) Compute $\sum_{n=1}^{N} \frac{1}{n(n+1)}$ and then $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

      HINT: Use the *partial fraction decomposition* $\frac{1}{(x-a)(x-b)} = \frac{1}{a-b}\left[\frac{1}{x-a} - \frac{1}{x-b}\right]$.

   b) Use this to estimate $\sum_{=n1}^{\infty} \frac{1}{n^2}$

   SOLUTION:

   (a) $\sum_{k=1}^{N} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} = 1 - \frac{1}{n+1} \to 1$ as $n \to \infty$.

   (b) $\sum_{n=1}^{\infty} \frac{1}{n^2} > \sum_{n=1}^{N} \frac{1}{n(n+1)} = 1$. Since $\sum_{n=2}^{\infty} \frac{1}{n^2} < \sum_{n=2}^{N} \frac{1}{n(n-1)} = 1$, $\sum_{n=1}^{\infty} \frac{1}{n^2} < 1+1 = 2$.

8. [#14.45] Let $S_n := a_1 + \cdots + a_n$. If $S_n = 1/n$ for all $n \geq 1$, find a_n for all $n \geq 1$.

   SOLUTION: $a_1 = S_1 = 1$. When $n \geq 2$, $a_n = S_n - S_{n-1} = \frac{1}{n} - \frac{1}{n-1} = \frac{-1}{n(n-1)}$.

[Last revised: October 26, 2013]