Math 202, Fall 2009 Jerry L. Kazdan

Homework Set 5 (Due in class on Thursday, Oct. 15)

(late papers accepted until 1:00 Friday)

Exam 1 will be held in class on Thursday, Oct. 22. You may use one 3×5 card with notes (on both sides). There will be no homework that week.

The problem numbers refer to the D'Angelo - West text.

1. A tennis ball is dropped from a height H. After each bounce it returns to two-thirds of its height on the previous bounce. How far does the ball travel until it is at rest on the floor?

   SOLUTION: Suppose the ball bounces back to height H_n after n-th bounce, then $H_1 = \frac{2}{3}H$ and $H_n = (\frac{2}{3})H_{n-1} = (\frac{2}{3})^2 H_{n-2} = \cdots = (\frac{2}{3})^n H$. So the total length that the ball travels is

$$H + 2\sum_{n=1}^{\infty} H_n = H + 2H\sum_{n=1}^{\infty}(\frac{2}{3})^n = H + 2H\frac{2/3}{1-2/3} = 5H.$$

2. [#14.2] For each condition below, give an example of an *unbounded* sequence such that $a_{n+1} - a_n > 0$ for all $n \in \mathbb{N}$ and the specified condition holds.

   a) $\lim (a_{n+1} - a_n) = 0$.

   b) $\lim (a_{n+1} - a_n)$ does not exist.

   c) $\lim (a_{n+1} - a_n) = L$, where $L > 0$.

   SOLUTION: (a) $a_n = \sqrt{n}$. (b) $a_n = n^2$. (c) $a_n = nL$.

3. Suppose that $x_0 = c$ for some real c and $x_{n+1} = \sqrt{1 + x_n^2}$ for all $n \in \mathbb{N}$. For which c does x_n converge?

   SOLUTION: It is clear that the x_k are increasing. So the convergence depends on if they are bounded. Here are two approaches.

   METHOD 1: Let $y_n := x_n^2$. Then $y_{n+1} = 1 + y_n$. The y_n's are clearly unbounded. Thus the x_n's diverge to infinity.

   METHOD 2: By contradiction, say $x_n \to L$. Then $L = \sqrt{1 + L^2} > L$, which is a contradiction.

4. [#14.9]. Proof or counterexample. Suppose that $x_n \to L$.

   a) For all $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $|x_{n+1} - x_n| < \epsilon$.

b) There exists $n \in \mathbb{N}$ such that for all $\epsilon > 0$, $|x_{n+1} - x_n| < \epsilon$.

c) There exists $\epsilon > 0$ such that for all $n \in \mathbb{N}$: $|x_{n+1} - x_n| < \epsilon$.

d) For all $n \in \mathbb{N}$ there exists $\epsilon > 0$ such that $|x_{n+1} - x_n| < \epsilon$.

SOLUTION: (a) True. For any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $|x_n - L| < \frac{1}{2}\varepsilon$ when $n \geq N$. Pick any $n \geq N$, we also have $n + 1 \geq N$, and thus $|x_{n+1} - L| < \frac{1}{2}\varepsilon$. By the triangle inequality, $|x_{n+1} - x_n| \leq |x_{n+1} - L| + |L - x_n| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$.

Equivalently, $x_{n+1} - x_n \to L - L = 0$

(b) False. Let $x_n = \frac{1}{n}$ and $L = 0$. Then for a *fixed* n the assertion $x_{n+1} - x_n = \frac{1}{n+1} - \frac{1}{n} < \varepsilon$ for *all* $\varepsilon > 0$ is certainly false. For a given $\varepsilon > 0$ it is true for all sufficiently large n, but this is supposed to be satisfied for some specific n.

(c) True. There is $N \in \mathbb{N}$ such that $|x_n - L| < 1$ when $n \geq N$. In particular, we have $|x_n| < L + 1$ when $n \geq N$. Now let $M = \max(|x_1|, |x_2|, \ldots, |x_{n-1}|, L + 1)$, then $|x_n| < M$ for any $n \in \mathbb{N}$. By triangle inequality, $|x_{n+1} - x_n| \leq |x_{n+1}| + |-x_n| \leq 2M$. Now we can finish the proof by putting $\varepsilon = 2M$.

(d) True. This follows from (c). Just pick the same ε as in (c).

Alternate wording: since any convergent sequence is bounded, then there is an M such that $|x_n| \leq M$ for all n. Thus $|x_{n+1} - x_n| \leq |x_{n+1}| + |x_n| \leq 2M$ for all n so the assertion is true with $\varepsilon = 2M$.

5. [#14.13] If $x_n \to L$, then every subsequence converges to L.

   SOLUTION: Let x_{k_n} be a subsequence of x_n, then $k_n \geq n$. For any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $|x_n - L| < \varepsilon$ if $n \geq N$. When $n \geq N$, $k_n \geq n \geq N$, so $|x_{k_n} - L| < \varepsilon$.

6. [#14.15] Let b and L be real numbers. If $b \leq L + \epsilon$ for all $\epsilon > 0$, prove that $b \leq L$.

   SOLUTION: By contradiction, assume $b > L$. Let $\varepsilon = \frac{b-L}{2}$. Then $b - L - \varepsilon = \frac{b-L}{2} > 0$. This contradicts $b \leq L + \varepsilon$.

7. If c is a complex number with $|c| < 1$, show that $(n^2+1)c^n \to 0$. Does $n^5 c^n$ converge?

   SOLUTION: We use the ratio test. Write $a_n = (n^2+1)c^n \to 0$. Then $\frac{|a_{n+1}|}{|a_n|} = \frac{n^2+2n+2}{n^2+1}|c|$ which converges to $|c| < 1$ as $n \to \infty$, so $\lim(n^2 + 1)c^n \to 0$. If $|c| \geq 1$, then $|a_n| \geq n^2 + 1$ so the sequence clearly diverges.

   If $|c| < 1$ the sequence $n^5 c^n \to 0$ for the same reason. It too diverges if $|c| \geq 1$

8. [#14.18] If $a_1 = 1$ and $a_{n+1} = \sqrt{3a_n + 4}$ for $n \geq 1$, show that $a_n < 4$ for all $n \geq 1$.

   SOLUTION: $a_1 = 1 < 4$. By a simple induction: if $a_k < 4$, then $a_{k+1} = \sqrt{3a_k + 4} < \sqrt{3 \cdot 4 + 4} = 4$.

9. [#14.32] A runaway train is hurtling toward a brick wall at a speed of 100 miles per hour. When it is 2 miles from the wall, a (speedy) fly begins to fly repeatedly between the train and the wall at the speed of 200 miles per hour. Determine how far the fly travels before it is smashed.

SOLUTION 1: Notation: $V_T = 100$ mph, $V_F = 200$ mph, $D_0 = 2$ miles is the initial distance between the train and the wall.

FIRST CYCLE: The fly begins at the wall and after time t_1 hours meets the train. In this time the fly has gone $V_F t_1$ miles and the train has gone $V_T t_1$. Since they have met, $V_F t_1 + V_T t_1 = D_0$. That is,

$$t_1 = \frac{D_0}{V_F + V_T} = \frac{2}{300}.$$

The fly returns back to the wall so total elapsed time is $2t_1$.The train is now $2t_1 V_T$ closer to the wall so its distance to the wall is

$$D_1 = D_0 - 2V_T t_1 = D_0 - 2\frac{V_T}{V_F + V_T} D_0 = cD_0, \quad \text{where} \quad c := 1 - 2\frac{V_T}{V_F + V_T} = \frac{1}{3}.$$

SECOND CYCLE: This is the same as for the first cycle only now the train is a distance D_1 from the wall. The time, t_2, for the fly to meet the train is

$$t_2 = \frac{D_1}{V_F + V_T} = \frac{cD_0}{V_F + V_T} = ct_1.$$

The fly returns to the wall so the total time elapsed on this cycle is $2t_2$, during which the train has gone $2t_2 V_T$ miles so its distance to the wall now is

$$D_2 = D_1 - 2t_1 V_T = cD_1.$$

k^{th} CYCLE: As before

$$t_k = ct_{k-1} = c^{k-1} t_1.$$

The total flying time T for all if the cycles is therefore

$$T = 2(t_1 + t_2 + \cdots) = 2(1 + c + c^2 + \cdots) t_1 = \frac{2t_1}{1 - c} = \frac{4}{300} \frac{1}{(1 - \frac{1}{3})} = \frac{1}{50} \text{ hours.}$$

so the total distance the fly has flown is $V_F T = 200/50 = 4$ miles.

SOLUTION 2: (Far simpler, but most people – including John von Neumann – give the previous solution). The train must travel 2 miles before it hits the wall. That takes $2/100 = 1/50$ hours. In this time the fly travels $200/50 = 4$ miles.

10. Show that $n^{\frac{1}{n}} \to 1$. [HINT: Let $x_n := n^{\frac{1}{n}} - 1$. Show that $x_n \to 0$ similarly to HW 4, Problem 7, by proving and using the inequality: if $a > 0$, then $(1+a)^n > 1 + na + \frac{n(n-1)}{2}a^2$].

SOLUTION: Let $a_n = n^{\frac{1}{n}} - 1 > 0$, then the assertion is equivalent to show that $a_n \to 0$ as $n \to \infty$. Since, by the binomial theorem (or induction), for any $c > 0$

$$(1+c)^n \geq 1 + nc + \frac{n(n-1)}{2}c^2 > \frac{n(n-1)}{2}c^2,$$

then

$$n = (1 + x_n)^n > \frac{n(n-1)}{2}x_n^2.$$

Hence, as $n \to \infty$

$$x_n^2 < \frac{n}{n(n-1)/2} = \frac{2}{n-1} \to 0.$$

[Last revised: October 26, 2013]