Math 202, Fall 2002 Jerry L. Kazdan

Homework Set 4 (Due in class on Thursday, Oct. 8)

(late papers accepted until 1:00 Friday)

The problem numbers refer to the D'Angelo - West text.

1. [# 13.20] For each set S below, obtain a sequence in S converging to $\sup(S)$ and a sequence converging to $\inf(S)$.

   a). $S = \{x \in \mathbb{R} : 0 \le x < 1\}$ b). $S = \{\frac{2+(-1)^n}{n},\ n \in \mathbb{N}\}$.

   SOLUTION: (a) $0 \to \inf(S) = 0$ and $1 - \frac{1}{n} \to \sup(S) = 1$.

   (b) $\frac{2+(-1)^n}{n} \to \inf(S) = 0$ and $\frac{3}{2} \to \sup(S) = \frac{3}{2}$.

2. [# 13.20] For each set $S \subset \mathbb{R}$ below, determine if it is bounded above and/or below, and if so, find $\inf(S)$ and $\sup(S)$ (if they exist):

   a). $S = \{x :\ x^2 < 5x\}$ b). $S = \{x :\ 2x^2 < x^3 + x\}$ c). $S = \{x :\ 4x^2 > x^3 + x\}$

   SOLUTION: (a) Put $f(x) = x^2 - 5x$, then $S = \{x :\ f(x) < 0\}$. Since $f(x) = x(x-5)$, $f(x) < 0$ when x and $x - 5$ have different signs, that is, when $x \in (0, 5)$. So $S = (0, 5)$, $\inf(S) = 0$ and $\sup(S) = 5$.

   (b) Put $f(x) = x^3 - 2x^2 + x$, then $S = \{x :\ f(x) > 0\}$. Since $f(x) = x(x-1)^2$, $f(x) > 0$ iff $x > 0$ and $x \ne 1$. So $S = (0, 1) \bigcup (1, \infty)$, $\inf(S) = 0$ and $\sup(S)$ does not exist.

   (c) Put $f(x) = x^3 - 4x^2 + x$, then $S = \{x :\ f(x) < 0\}$.
   Since $f(x) = x(x-2-\sqrt{3})(x-2+\sqrt{3})$, then $f(x) < 0$ iff $2-\sqrt{3} < x < 2+\sqrt{3}$ or $x < 0$. So $S = (-\infty, 0) \bigcup (2-\sqrt{3},\ 2+\sqrt{3})$. Thus $\inf(S)$ does not exist and $\sup(S) = 2 + \sqrt{3}$.

3. [# 13.24] let $f,\ g : \mathbb{R} \to \mathbb{R}$ be bounded functions such that $f(x) \le g(x)$ for all x. Let F denote the image of f and G the image of g. Give examples (with pictures) of pairs of such functions with:

   a). $\sup(F) < \inf(G)$ b). $\sup(F) = \inf(G)$ c). $\sup(F) > \inf(G)$

   SOLUTION: (Sorry. No pictures.) (a) $f(x) = 0$ and $g(x) = 1$. (b) $f(x) = g(x) = 0$. (c) $f(x) = \sin(x)$ and $g(x) = \sin(x) + \frac{1}{2}$.

4. [# 13.27] Let $a_n = \sqrt{n^2 + n} = -n$. Compute $\lim_{n\to\infty} a_n$.

SOLUTION:

$$\begin{aligned} a_n =& (\sqrt{n^2+n} - n)\left(\frac{\sqrt{n^2+n}\ + n}{\sqrt{n^2+n}\ + n}\right) \\ =& \frac{n}{\sqrt{n^2+n}\ + n} = \frac{1}{\sqrt{1+\frac{1}{n}}\ + 1} \to \frac{1}{2} \quad \text{as} \quad n \to \infty \end{aligned}$$

5. [# 13.30] Let $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$. Show that $\lim_{n\to\infty}$ exists. [REMARK: In fact, the limit equals $\ln 2$ but this is not needed for this exercise.]

SOLUTION: We show this sequence is bounded above and monotonic increasing.

$$x_n \le \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} = n \cdot \frac{1}{n} = 1,$$

so x_n is bounded above. It is monotonic increasing because

$$x_{n+1} - x_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{2n+1} - \frac{1}{2n+2} > 0.$$

Therefore $\lim_{n\to\infty} x_n$ exists.

6. [# 13.32] *Nested Interval Property.* Let $\{I_n \subset \mathbb{R}\}$ be a sequence of closed (non-empty) intervals with I_n having length d_n such that $I_{n+1} \subseteq I_n$ and $d_n \to 0$. The Nested Interval Property states that for such a sequence there is exactly one point that belongs to all of the I_n.

a) Show that our Completeness Axiom implies the Nested Interval Property.

b) Show that the Nested Interval Property implies our Completeness Axiom.

SOLUTION: a). Suppose we have nested intervals $I_n = [a_n, b_n]$ such that

1. $I_{n+1} \subset I_n$,
2. $d_n = b_n - a_n$ converges to 0 as $n \to \infty$.

Let $A = \{a_n\}_{n\in\mathbb{N}}$. For any $m, n \in \mathbb{N}$, $a_m \le a_n \le b_n$ when $m < n$, and $a_m \le b_m \le b_n$ when $m \ge n$. So each b_n is an upper bound of A. The left end points, a_n are a bounded increasing sequence and hence converge to some limit α. Similarly, the right end points, b_n, are a bounded decreasing sequence and hence converge to some limit $\beta > \alpha$. Therefore $\bigcap I_n = \{\alpha \le x \le \beta\}$.

Now we use that the length of the intervals, $d_n = b_n - a_n$, converges to 0. Note $a_n \le \alpha \le \beta \le b_n$. Then, by the Squeeze Theorem $\alpha = \beta$ so $\bigcap I_n = \{\alpha\}$.

b). Suppose that S is set bounded above by M. Pick any $a_0 \in S$. If by luck a_0 is an upper bound of S, then a_0 is the least upper bound of S. Otherwise a_0 is not an upper bound of S, let $b_0 = M$, and then construct a sequence of nested intervals $I_n = [a_n, b_n]$ defined recursively as follows:

1. $I_0 = [a_0, b_0]$.
2. Say you have I_n. Let $c_n = \frac{a_n + b_n}{2}$ be its mid-point. If c_n is an upper bound of S, then let $a_{n+1} = a_n$ and $b_{n+1} = c_n$. Otherwise, let $a_{n+1} = c_n$ and $b_{n+1} = b_n$.

The intervals I_n are clearly nested and Length $(I_n) = b_n - a_n = \left(\frac{1}{2}\right)^n (b_0 - a_0) \to 0$

It is easy to show (using induction) that for every $n \in \mathbb{N}$, b_n is an upper bound of S while a_n is not. Also $d_n = b_n - a_n = (\frac{1}{2})^n d_0 \to 0$ as $n \to \infty$. By the Nested Interval Property, there is a unique element c in all of I_n.

To show that c is the least upper bound, it is clearly an upper bound since it is the limit of the b_n's, all of which are upper bounds. It is the *least* upper bound since any upper bound γ must satisfy $\gamma \geq a_n$ for all n. Thus, the least upper bound must be in all if the I_n's.

7. If $c > 0$, show that $\lim_{n\to\infty} c^{\frac{1}{n}} = 1$. [Suggestion: If $c > 1$, let $x_n := c^{\frac{1}{n}} - 1$ and show that $x_n \to 0$. Note $x_n > 0$ so use $c = (1 + x_n)^n \geq 1 + nx_n$. If $0 < c < 1$, take reciprocals.]

   Solution: This is trivial when $c = 1$.

   Case 1: $c > 1$. Let $a_n = c^{\frac{1}{n}} - 1 > 0$, then it suffices to show that $\lim_{n\to\infty} a_n = 0$. Since $c = (a_n + 1)^n \geq 1 + na_n$, then $a_n \leq \frac{c-1}{n} \to 0$ as $n \to \infty$. Because $a_n > 0$, then $\lim_{n\to\infty} a_n = 0$ by the squeeze theorem.

   Case 2: $0 < c < 1$, $\lim_{n\to\infty} c^{\frac{1}{n}} = \frac{1}{(\frac{1}{c})^{\frac{1}{n}}}$. Since $\frac{1}{c} > 1$, $\lim_{n\to\infty} (\frac{1}{c})^{\frac{1}{n}} = 1$, and hence $\lim_{n\to\infty} c^{\frac{1}{n}} = 1$.

[Last revised: October 19, 2013]