Homework Set 4 (Due in class on Thursday, Oct. 8)

(late papers accepted until 1:00 Friday)

The problem numbers refer to the D'Angelo - West text.

1. [# 13.20] For each set S below, obtain a sequence in S converging to $\sup(S)$ and a sequence converging to $\inf(S)$.

   a). $S = \{x \in \mathbb{R} : 0 \le x < 1\}$ b). $S = \{\frac{2+(-1)^n}{n},\ n \in \mathbb{N}\}$.

2. [# 13.20] For each set $S \subset \mathbb{R}$ below, determine if it is bounded above and/or below, and if so, find $\inf(S)$ and $\sup(S)$ (if they exist):

   a). $S = \{x :\ x^2 < 5x\}$ b). $S = \{x :\ 2x^2 < x^3 + x\}$ c). $S = \{x :\ 4x^2 > x^3 + x\}$

3. [# 13.24] let $f,\ g : \mathbb{R} \to \mathbb{R}$ be bounded functions such that $f(x) \le g(x)$ for all x. Let F denote the image of f and G the image of g. Give examples (with pictures) of pairs of such functions with:

   a). $\sup(F) < \inf(G)$ b). $\sup(F) = \inf(G)$ c). $\sup(F) > \inf(G)$

4. [# 13.27] Let $a_n = \sqrt{n^2 + n} - n$. Compute $\lim a_n$.

5. [# 13.30] Let $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$. Show that $\lim_{n\to\infty}$ exists. [Remark: In fact, the limit equals $\ln 2$ but that is not needed for this exercise.]

6. [# 13.32] *Nested Interval Property.* Let $\{I_n \subset \mathbb{R}\}$ be a sequence of closed (non-empty) intervals with I_n having length d_n such that $I_{n+1} \subseteq I_n$ and $d_n \to 0$. The Nested Interval Property states that for such a sequence there is exactly one point that belongs to all of the I_n.

   a) Show that our Completeness Axiom implies the Nested Interval Property.

   b) Show that the Nested Interval Property implies our Completeness Axiom.

7. If $c > 0$, show that $\lim_{n\to\infty} c^{\frac{1}{n}} = 1$. [Suggestion: If $c > 1$, let $x_n := c^{\frac{1}{n}} - 1$ and show that $x_n \to 0$. Note $x_n > 0$ so use $c = (1 + x_n)^n \ge 1 + nx_n$. If $0 < c < 1$, take reciprocals.]

[Last revised: October 6, 2009]