Math 202, Fall 2009 Jerry L. Kazdan

Homework Set 3 (Due in class on Thursday, Oct. 1)

(late papers accepted until 1:00 Friday)

The problem numbers refer to the D'Angelo - West text.

1. a) If a and b are rational numbers, show that the set S of real numbers of the form $a + b\sqrt{7}$ form a field. Note that since S is a subset of the real numbers, it automatically inherits many of the properties of a field. Consequently, you need only show the following:
   i) If z and w are in S, then so are $z + w$ and zw. So the set S is *closed* under both addition and multiplication.
   ii) The elements in S have additive inverses in S.
   iii) The elements in S have multiplicative inverses in S.

   b) Does the set T of real numbers of the form $a + b\sqrt{6}$ form a field? (Again, a and b are rational numbers).

   SOLUTION: (a) For any $z = a + b\sqrt{7} \in S$ and $w = c + d\sqrt{7} \in S$ where a, b, c, d are rational, we have $z + w = (a + c) = (b + d)\sqrt{7} \in S$ and $zw = (ac + 7bd) + (ad + bc)\sqrt{7}$. The additive inverse of z is $-z = (-a) + (-b)\sqrt{7}$. Since $\sqrt{7}$ in irrational, $\sqrt{7} \neq \frac{a}{b}$, and hence $a \neq b\sqrt{7}$. The multiplicative inverse of z is

$$\frac{1}{a + b\sqrt{7}} = \frac{1}{a + b\sqrt{7}}\left(\frac{a - b\sqrt{7}}{a - b\sqrt{7}}\right) = \frac{a - b\sqrt{7}}{a^2 - 7b^2} = \frac{a}{a^2 - 7b^2} + \frac{-b}{a^2 - 7b^2}\sqrt{7}.$$

   (b) Since $\sqrt{6}$ is irrational, T is a field. The proof is very similar to part a).

2. If $x > 0$, use induction to verify the inequality $(1 + x)^k \geq 1 + kx$ for any integer $k = 1, 2, \ldots$.

   SOLUTION: $(1 + x)^1 = 1 + 1 \cdot x$. If $(1 + x)^n \geq 1 + nx$ for some $n \in \mathbb{N}$, then

$$\begin{aligned}(1 + x)^{n+1} &= (1 + x)^n(1 + x) \geq (1 + nx)(1 + x) \\ &= 1 + (n + 1)x + nx^2 \geq 1 + (n + 1)x.\end{aligned}$$

   Thus $(1 + x)^k \geq 1 + kx$ for any $k \in \mathbb{N}$.

3. A standard example of using induction is to verify the formula $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$. It is much less well-known that you can use induction to *discover* this

formula. Here is the procedure. Let $S_n := 1+2+3+\cdots+n$. Then the key (induction) step is

$$S_{n+1} - S_n = n+1 \quad \text{(why?)} \qquad \text{with the } \textit{initial condition} \quad S_1 = 1. \tag{1}$$

We want to solve this *difference equation* (1) directly. Experimenting, we find $S_2 = 3$, $S_3 = 6$, $S_4 = 10$, $S_5 = 15$. This leads us to guess there might be a formula of the form

$$S_n = an^2 + bn + c,$$

where the coefficients a, b, and c are still unknown. To find them we plug this into (1) and hope we will succeed in finding them. Do this. You will succeed.

It could be that equation (1) has several solutions. But it doesn't. Show that if $\hat{S}_n$ also satisfies (1), then $\hat{S}_n = S_n$. [MORAL: If you have found a solution, then you have found the unique solution.]

SOLUTION: To prove $\hat{S}_n = S_n$, Let $Q_n = \hat{S}_n - S_n$ and note that $Q_1 = 1 - 1 = 0$. We use induction to show that $Q_n = 0$ for all n. Note that by (1)

$$Q_{n+1} - Q_n = (n+1) - (n+1) = 0.$$

Thus $Q_{n+1} = Q_n$ for all n. Since $Q_1 = 0$, by a trivial induction $Q_n = 0$ for all integers $n \in \mathbb{N}$.

4. Use the idea of the previous problem to find a formula for $T_n := 1^2 + 2^2 + 3^2 + \cdots + n^2$.

SOLUTION: We seek a $T_n = an^3 + bn^2 + cn + d$ such that $T_{n+1} - T_n = (n+1)^2$ and $T_1 = 1$. The argument in the previous problem shows that if we can find such a T_n, then it is the *only* possible solution (so T_n is *uniquely determined*). We compute:

$$\begin{aligned} T_{n+1} - T_n =& a(n+1)^3 + b(n+1)^2 + c(n+1) + d - (an^3 + bn^2 + cn + d) \\ =& 3an^2 + (3a + 2b)n + (a + b + c). \end{aligned}$$

Now we want

$$T_{n+1} - T_n = (n+1)^2 = n^2 + 2n + 1 \qquad \text{and} \qquad T_1 = 1.$$

Comparing these, the unknown coefficients a, b, c, and d must satisfy:

$$3a = 1, \qquad 3a + 2b = 2, \qquad a + b + c = 1.$$

Consequently, $a = 1/3$, $b = 1/2$, and $c = 1/6$. Using these and $T_1 = 1$ we find $d = 0$. Therefore $T_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$.

5. Let a_n be a sequence of real numbers with the property that the "even" sub-sequence a_{2n} converges to A and the "odd" sub-sequence a_{2n+1} converges to B [Baby Example: $a_n = (-1)^n$]. If the original sequence a_n converges, show that $A = B$ (so the baby example does **not** converge).

SOLUTION: Method 1, Suppose a_n converges to C. We use that then $a_{k+1} - a_k \to 0$. Say k is even, so $k = 2n$. Then $B - A = \lim_{n\to\infty}[a_{2n+1} - a_{2n}] = 0$.

Method 2 – using ε: For any $\varepsilon > 0$, there are $N_1, N_2, N_3 \in \mathbb{N}$ such that when $n \ge N_1$, then $|a_{2n+1} - A| < \frac{1}{4}\varepsilon$, when $n \ge N_2$, then $|a_{2n} - B| < \frac{1}{4}\varepsilon$, and when $n \ge N_3$, then $|a_n - C| < \frac{1}{4}\varepsilon$. Let $N = \max(N_1, N_2, N_3)$. Then for any $n \ge N$, by the triangle inequality

$$|A - B| \le |A - a_{2n+1}| + |a_{2n+1} - C| + |C - a_{2n}| + |a_{2n} - B| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $|A - B| = 0$, that is, $A = B$.

6. [# 13.9] Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) := \dfrac{2x - 8}{x^2 - 8x + 17}$. Then the supremum of the image of f is 1. Give a proof or counterexample.

SOLUTION: Idea: rewrite f as $f(x) = \frac{2(x-4)}{(x-4)^2+1}$ and let $t = x - 4$. Then we need to show that $2t/(t^2 + 1) \le 1$. This is immediate by the AM-GM inequality. So the supremum of the image is at most 1. If we let $t = 1$, that is, $x = 5$, we have $f(5) = 1$, so the supremum of the image of f is 1.

7. Show that $\displaystyle\lim_{n\to\infty} \frac{10^n}{n!} = 0$.

SOLUTION: Ratio test: let $a_n = \dfrac{10^n}{n!}$; then $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{10}{n+1} \to 0 < 1$ as $n \to \infty$.

8. [# 13.34]

   a) Given any two rational numbers $r < s$, prove there is an *irrational* number c between them: $r < c < s$.

   b) Given any two real numbers $a < b$, prove there is a *rational* number r between them: $a < r < b$.

SOLUTION: (a). Choose an irrational number α that is less than the distance between r and s, say $\alpha = \frac{\sqrt{2}}{2}(s - r)$. Then put $c = r + \alpha < s$.

(b) Idea: This is clear if $b - a > 1$ since then there is at least one integer N between a and b (Proof: let k be the largest integer less than a. Then $N = k + 1$ satisfies $a < N < b$).

To reduce to this case, pick $n \in \mathbb{N}$ such that $n(b-a) > 1$. Then $nb - na > 1$ so there must be an integer, say k, between na and nb: $na < k < nb$. Thus $a < k/n < b$ so the desired rational number is k/n.

9. [#13.11] Suppose the sequences a_n and b_n of real numbers both converge. For each of the following assertions give a proof or counterexample,

   a) If $\lim_{n\to\infty} a_n < \lim_{n\to\infty} b_n$, then there exists $N \in \mathbb{N}$ such that $n \geq N$ implies that $a_n < b_n$.

   b) If $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$, then there exists $N \in \mathbb{N}$ such that $n \geq N$ implies that $a_n \leq b_n$.

   SOLUTION: (a) True. Write $A = \lim_{n\to\infty} a_n$ and $B = \lim_{n\to\infty} b_n$, then $A < B$. There is $N \in \mathbb{N}$ such that if $n > N$ then both $|a_n - A| < \frac{B-A}{2}$ and $|b_n - B| < \frac{B-A}{2}$. Hence

$$a_n < A + \tfrac{1}{2}(B-A) = B - \tfrac{1}{2}(B-A) < b_n.$$

   (b) False. Counterexample: Take $a_n = \frac{1}{n}$ and $b_n = 0$. Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$ but $a_n > b_n$.

   Another example: with $a_n >$ as above, let $b_n = -a_n < 0$.

[Last revised: October 19, 2013]