Math 202 Jerry L. Kazdan

Homework Set 2 (Due in class on Thursday, Sept. 25)

(late papers accepted until 1:00 Friday)

The problem numbers refer to the D'Angelo - West text.

1. [# 1.8] In the morning section of a calculus course, 2 of the 9 women and 2 of the 10 men received the grade of A. In the afternoon section 6 of the 9 womem and 9 of the 14 men received an A. Verify that in each section, a higher proportion of women than men received an A – but that in the combined course a lower portion of women than men received a grade of A. Explain!

   SOLUTION: People in the afternoon section performed much better than people in the afternoon did. Even though there are higher proportion of women than men received an A in each section, the difference is small compared to the difference between sections. At the end, a lower portion of women than men received a grade of A because there are lower portion of women than men in the afternoon section. (This is called the Simpson's paradox.)

2. [# 1.20] Suppose that r and s are distinct real solutions of the quadratic equation $ax^2 + bx + c = 0$. Express $r + s$ and rs in terms of a, b, and c.

   SOLUTION: Since r and s are distinct real solutions of $ax^2+bx+c=0$, $ax^2+bx+c = a(x-r)(x-s) = ax^2 - a(r+s)x + ars$. So we have $-a(r+s) = b$ and $ars = c$, which implies that $r + s = -\frac{b}{a}$ and that $rs = \frac{c}{a}$.

   ALTERNATIVE SOLUTION:

$$r + s = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{a}.$$

$$rs = \frac{-b + \sqrt{b^2 - 4ac}}{2a}\frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{c}{a}.$$

   Note that the first solution does not rely on knowing an explicit formula for the solution and is thus preferable. It works for a polynomial of any degree – even though there is no algebraic formula for the roots of a general polynomial of degree greater than 4 (as an example, see the next problem).

3. Suppose that r, s, and t are distinct real solutions of the cubic equation $ax^3 + bx^2 + cx + d = 0$. Express $r + s + t$ and rst in terms of a, b, c. and d. [For this you don't need the formula for the roots of cubic].

SOLUTION: $ax^3 + bx^2 + cx + d = a(x-r)(x-s)(x-t) = ax^3 - a(r+s+t)x^2 + a(rs+rt+st)x - arst$. So we have $-a(r+s+t) = b$ and $-arst = d$, which implies that $r+s+t = -\frac{b}{a}$ and that $rst = -\frac{d}{a}$.

4. [# 1.22] You have two identical glasses. The first has with a cup of wine, the second has a cup of water. You remove 1 teaspoon of wine from the first glass and add it to the second. Stir carefully. Then take one teaspoon of the mixture from the second glass and add to the first. Prove that the amount of wine in the first glass now equals the amount of water in the second glass.

   SOLUTION: Let a be the volume of wine in the first glass, b be the volume of water in the first glass, and c be the volume of water in the second glass, all measured at the end of all the operations. Since we removed one teaspoon of liquid from the first glass and then put back the same amount, there should still be a cup of water in the first glass at the end, and thus $a+b$ is equal to one cup. Since there is one cup of water in total, $b+c$ is also equal to one cup. Thus $a+b = b+c$, which implies that $a = c$, that is, the amount of wine in the first glass equals the amount of water in the second glass.

5. [# 1.30] Let x, y, u, and v be real numbers.
   a) Prove that $(xu+yv)^2 \leq (x^2+y^2)(u^2+v^2)$.
   b) Determine precisely when equality holds in this.

   SOLUTION: (a)

$$\begin{aligned}
&(x^2+y^2)(u^2+v^2) - (xu+yv)^2 \\
&= x^2u^2 + x^2v^2 + y^2u^2 + y^2v^2 - (x^2u^2 + 2xuyv + y^2v^2) \\
&= x^2v^2 + y^2u^2 - 2xuyv \\
&= (xv - yu)^2 \geq 0
\end{aligned}$$

   Hence $(xu+yv)^2 \leq (x^2+y^2)(u^2+v^2)$. (b) The equality holds if and only if $xv = yu$.

6. [# 1.46] Determine the images of the functions $f : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$a).\ f(x) := \frac{x^2}{1+x^2} \qquad\qquad b).\ f(x) := \frac{x}{1+|x|}$$

   SOLUTION: (a) $f(0) = 0$. f is monotonically increasing on $[0,\infty)$, converging to 1 as x goes to ∞. Hence $f([0,\infty)) = [0,1)$. Since f is even, $f((-\infty,0]) = f([0,\infty)) = [0,1)$. Thus the image of f is $[0,1)$.

   (b) f is monotonically increasing on $(-\infty,\infty)$, converging to -1 as x goes to $-\infty$ and converging to 1 as x goes to ∞. Thus the image of f is $(-1,1)$.

7. [# 1.49] Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. For their sum and product, determine which statements below are True. If True, please give a proof. If False, give a counterexample.

   a) If f and g are bounded, then $f+g$ is bounded.

   b) If f and g are bounded, then fg is bounded.

   c) If $f+g$ is bounded, then f and g are bounded.

   d) If fg is bounded, then f and g are bounded.

   e) If both $f+g$ and fg are bounded, then f and g are bounded.

   SOLUTION: (a), and (b) are true – and obvious.

   (c) $f(x) = x$ and $g(x) = -x$ are not bounded but $f+g = 0$ is bounded.

   (d) Let $f(x) = x^2$ and $g(x) = 1/(1+x^2)$. f is not bounded but $0 \le fg < 1$ is bounded. Another example: $f(x) = e^x$, $g(x) = e^{-x}$.

   (e) True. Use $f^2 \le f^2 + g^2 = (f+g)^2 - 2fg$. Both terms on the right side are bounded. There is an identical bound for g^2.

8. [# 1.50] NOTATION: If S is a subset of the domain of a function f, write $f(S) := \{f(x) : x \in S\}$.

   Let C and D be subsets of the domain of f.

   a) Prove that $f(C \cap D) \subseteq f(C) \cap f(D)$.

   b) Give an example where equality does not hold in the above.

   SOLUTION: (a) If $y \in f(C \cap D)$, then $y = f(x)$ for some $x \in C \cap D$. Since $x \in C \cap D \subset C$, $y = f(x) \in f(C)$. Similarly, we have $y \in f(D)$. Thus $y \in f(C) \cap f(D)$.

   (b) Let $f(x) = x^2$ so the domain of f is all the real numbers. Let $C = (1,0)$ and $D = (0,1)$, then $C \cap D = \emptyset$ so $f(C \cap D) = \emptyset$ while $f(C) \cap f(D) = (0,1)$. Another example: still with $f(x) = x^2$ but $C = (-1,\ 0]$ and $D = [0,\ 1)$. Here $C \cap D = \{0\}$.

9. [# 1.52] Let M and N be non-negative real numbers. Suppose that $|x+y| \le M$ and $|xy| \le N$. Determine the maximum possible value of x as a function of M and N.

   SOLUTION: This is similar to #7e. By looking at the graph of these regions it is pretty clear that the maximum possible x is achieved at an intersection point of $x+y = M$ and $xy = -N$. Namely, $(x,y) = (\frac{M}{2} + \sqrt{N + \frac{M^2}{4}}, \frac{M}{2} - \sqrt{N + \frac{M^2}{4}})$. For a rigorous proof, we will use $x+y < M$ and $xy > -N$ to show that $x \le \frac{M}{2} + \sqrt{N + \frac{M^2}{4}}$.

   Assume that $x > 0$. Since $x+y \le M$, $y \le M-x$. Hence we have $-N \le xy \le x(M-x)$. So $x^2 - Mx - N \le 0$, that is, $(x - \frac{M}{2})^2 \le N + \frac{M^2}{4}$. So $x \le \frac{M}{2} + \sqrt{N + \frac{M^2}{4}}$.

10. [# 1.55] Let $\mathbb{F}$ be a field consisting of exactly three elements: 0, 1, and x. Prove that $1+1=x$ and that $x \cdot x = 1$. Obtain the addition and multiplication table for $\mathbb{F}$.

SOLUTION: $x+1$ can be x, 1 or 0. If $x+1=x$, then $1=0$, but $1 \neq 0$. If $x+1=1$, then $x=0$, but $x \neq 0$. So $x+1=0$.

$x+x$ can be x, 1 or 0. If $x+x=x$, then $x=0$, but $x \neq 0$. If $x+x=0$, then $x+x=x+1$ and hence $x=1$, but $x \neq 1$. So $x+x=1$. We can get $1+1=x$ similarly.

The addition table is as following.

$+$	0	1	x
0	0	1	x
1	1	x	0
x	x	0	1

Multiplication: $x \cdot x$ can be x, 1 or 0. If $x \cdot x = 0$, then $x=0$, but $x \neq 0$. If $x \cdot x = x$, then $x=1$, but $x \neq 1$. So $x \cdot x = 1$. The multiplication table is

$\cdot$	0	1	x
0	0	0	0
1	0	1	x
x	0	x	1

[The following problem used to be the last part of #12]

11. If 3^{1025} is divided by 5, what is the remainder?

SOLUTION: $3^2 \equiv 4 \mod 5$. $3^3 \equiv 4 \cdot 3 \equiv 2 \mod 5$. $3^4 \equiv 2 * 3 \equiv 1 \mod 5$. Hence $3^{1025} = 3^{1024} \cdot 3 = (3^4)^{256} \cdot 3 \equiv 1^{256} \cdot 3 \equiv 3 \mod 5$.

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[The following two problems are now **deleted** from this assignment.]

A. [# 1.56] Is there a field with exactly four elements? Is there a field with exactly six elements?

B. Let n be a positive integer. For any integers a, b we say that a *equals* b *mod* n if a and b have the same remainders when divided by n (equivalently, if $b-a$ is divisible by n). We write: $a \equiv b \pmod{n}$. So modulo n the possible remainders are $0, 1, 2, \ldots, (n-1)$ and every integer is equivalent to one of these. We tell time modulo 24 hours.

a) If $a \equiv r \pmod{n}$ and $b \equiv s \pmod{n}$, show that $a+b \equiv r+s \pmod{n}$ and $ab \equiv rs \pmod{n}$.

b) Show that the integers mod 5 form a field.

c) If p is a prime number, show that the integers mod p form a field.

[The only place cleverness is needed is showing that each non-zero element has a multiplicative inverse. There is no simple formula for this. One approach to show that k $(k \neq 0)$ has a multiplicative inverse is to first show that the numbers k, $2k$, $3k, \ldots, (p-1)k$ are all distinct mod p. Then, why must they be the integers 1, 2,..., $p-1$ only rearranged? Why does this show that k has a multiplicative inverse?]

d) If 3^{1025} is divided by 5, what is the remainder?

[Last revised: September 18, 2013]